What is Extinction?

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Extinction is ubiquitous in nature

- **Astronomy: scattering by interstellar dust**

  E.M.Purcell, C.R.Pennypacker, Scattering and absorption of light by nonspherical dielectric grains, *Astrophysical Journal* **186**, 705, 1973
  2200 citations in Google Scholar

  B.T.Draine, The discrete-dipole approximation and its application to interstellar graphite grains, *Astrophysical Journal* **333**, 848, 1988
  1800 citations in Google Scholar

- **Atmospheric optics, clouds, climate**

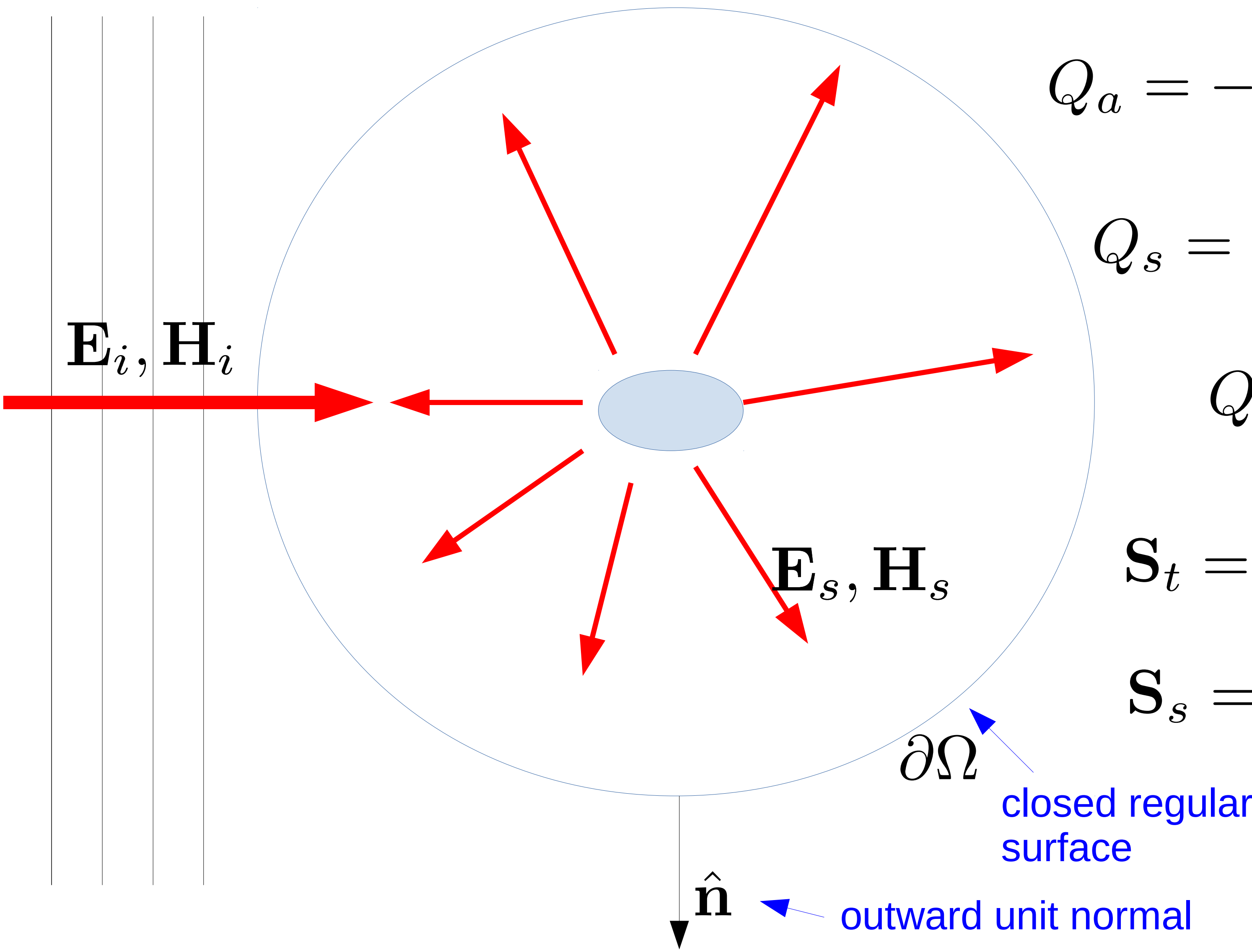
- **Biomedical optics and imaging**

- **Neuron scattering**

- **But the very notion of extinction has long been riddled with paradoxes**

Some Definitions (single particle)

$$\mathbf{E}_t = \mathbf{E}_i + \mathbf{E}_s \ , \quad \mathbf{H}_t = \mathbf{H}_i + \mathbf{H}_s$$



$$Q_a = -\oint_{\partial\Omega} [\mathbf{S}_t(\mathbf{r}) \cdot \hat{\mathbf{n}}] d^2 r$$

$$Q_s = \oint_{\partial\Omega} [\mathbf{S}_s(\mathbf{r}) \cdot \hat{\mathbf{n}}] d^2 r$$

$$Q_e = Q_a + Q_s$$

$$\mathbf{S}_t = \frac{c}{4\pi} \langle \mathbf{E}_t \times \mathbf{H}_t \rangle_{\text{time}}$$

$$\mathbf{S}_s = \frac{c}{4\pi} \langle \mathbf{E}_s \times \mathbf{H}_s \rangle_{\text{time}}$$

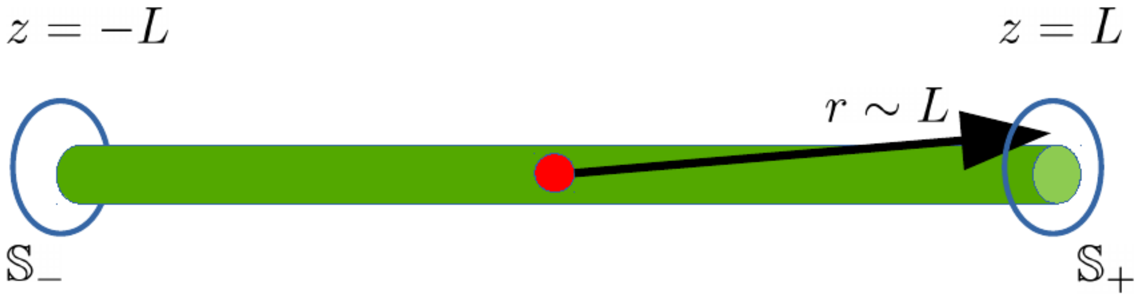
Extinction is an interference effect

$$Q_e = -\frac{c}{4\pi}\oint_{\partial\Omega}[\langle \mathbf{E}_i \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_s + \mathbf{E}_s \times \mathbf{H}_i \rangle_{\text{time}} \cdot \hat{\mathbf{n}}]d^2r$$

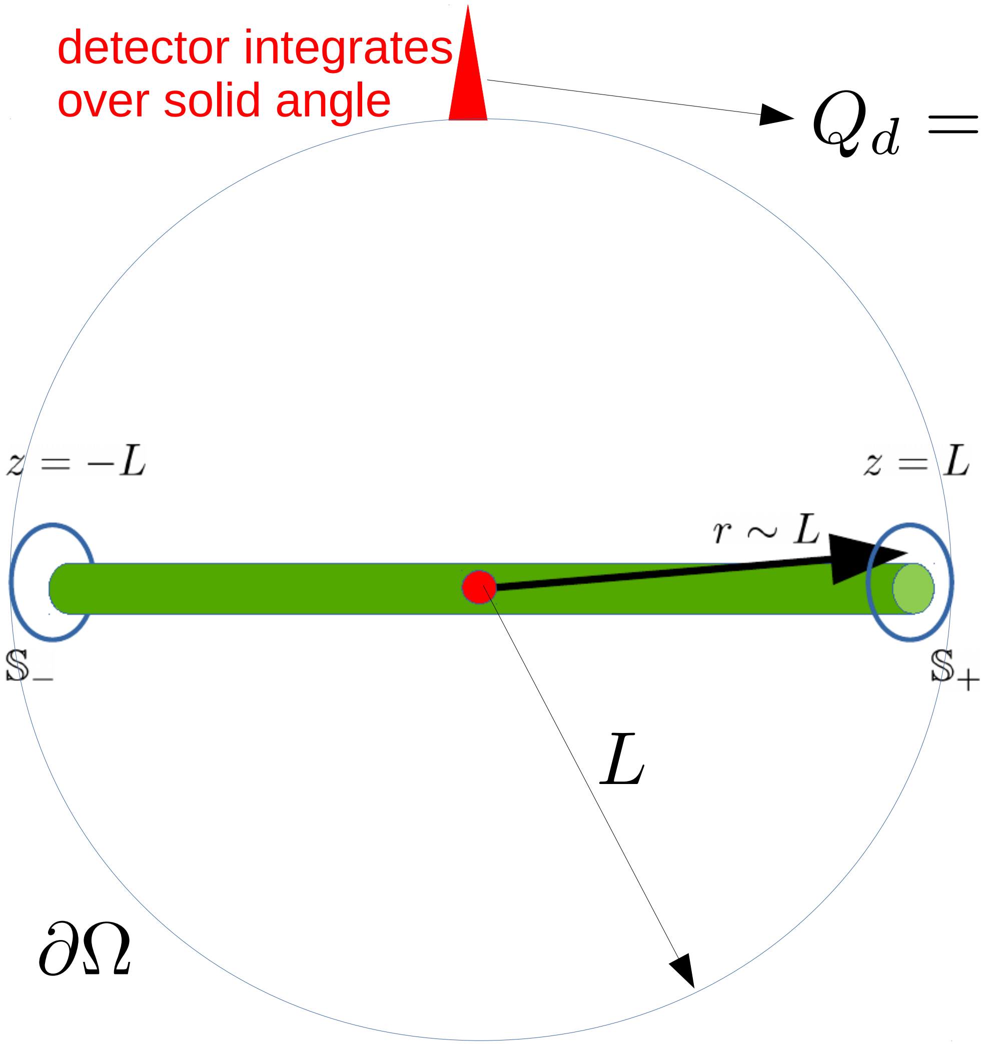
$$= -\frac{c}{4\pi}\oint_{\partial\Omega}[\langle \mathbf{E}_i \times \mathbf{H}_s + \mathbf{E}_s \times \mathbf{H}_i \rangle_{\text{time}} \cdot \hat{\mathbf{n}}]d^2r$$

How do we measure extinction (how is it affecting measurements)?

a) Let us try to separate scattered and incident fields by using a collimated beam (not necessarily relevant to astronomy, atmosphere, etc.). But then we would arrive at an apparent paradox:



Paradox 1: Extinction of collimated beams



$$Q_d = \oint_{\partial\Omega \setminus (\mathbb{S}_+ \cup \mathbb{S}_-)} [\mathbf{S}_t \cdot \hat{\mathbf{n}}] d^2 r \xrightarrow[L\to\infty]{} \oint_{\partial\Omega} [\mathbf{S}_s \cdot \hat{\mathbf{n}}] d^2 r = Q_s$$

$$Q_e = \oint_{\mathbb{S}_-} [\mathbf{S}_{\rm int} \cdot \hat{\mathbf{z}}] d^2 r - - \oint_{\mathbb{S}_+} [\mathbf{S}_{\rm int} \cdot \hat{\mathbf{z}}] d^2 r$$

$$L \to \infty \ : \ E_i, H_i \propto \text{const}$$
$$E_s, H_s \propto 1/L$$

$\Rightarrow Q_a < 0$?

$$Q_e \propto S E_i H_s \propto S E_0^2 / L \to 0$$

A resolution?

A pencil beam is an idealization; the beam must diverge

This is actually not enough; we can make a Gaussian beam arbitrarily well collimated by adjusting its waist.

If correct, that would mean that the scattered (and consequently extinguished) powers are not measurable.

Consideration complicated by the complex structure of vector Gauss beams. But, we have approximately.

Optical frequencies (500nm), waist 1mm --- beam would propagate without noticeable divergence for ~10meters – more than enough to demonstrate the paradox.

We will resolve the paradox for scalar waves, wherein it is conceptually the same. For vectorial EM waves the resolution is similar but technical details are more involved.

Scalar waves (wave function in QM scattering or acoustic scattering)

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-i\omega t}$$

$$\psi_t(\mathbf{r}) = \psi_i(\mathbf{r}) + \psi_s(\mathbf{r})$$

Wave function: $\Psi(\mathbf{r},t)$

Pressure field: $\mathrm{Re}[\Psi(\mathbf{r},t)]$

For monochromatic fields:

$$Q_a = -\oint_{\partial\Omega} [\mathbf{j}_t \cdot \hat{\mathbf{n}}] d^2 r$$

$$\mathbf{j}_t(\mathbf{r}) = \mathrm{Im}[\psi_t^*(\mathbf{r})\nabla\psi_t(\mathbf{r})]$$

total current (of energy, probability, etc.)

$$Q_s = \oint_{\partial\Omega} [\mathbf{j}_s \cdot \hat{\mathbf{n}}] d^2 r$$

$$\mathbf{j}_s(\mathbf{r}) = \mathrm{Im}[\psi_s^*(\mathbf{r})\nabla\psi_s(\mathbf{r})]$$

scattered current

$$Q_e = Q_a + Q_s = -\oint_{\partial\Omega} [\mathbf{j}_e \cdot \hat{\mathbf{n}}] d^2 r$$

$$\mathbf{j}_e = \mathrm{Im}(\psi_i^*\nabla\psi_s + \psi_s^*\nabla\psi_i)$$

interference term in the definition of extinction

in the “interaction region”

$$\psi_s(\mathbf{r}) = \alpha\psi_i(0)\frac{e^{ikr}}{r} \propto 1/L$$

Paradox is still present

Scalar Gaussian beam; Choose

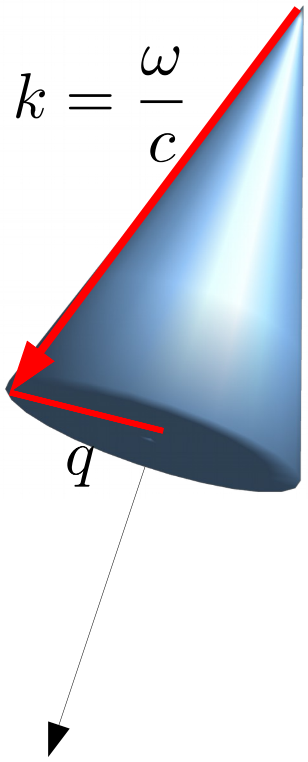
$$\psi_i(\boldsymbol{\rho}, z) = \frac{1}{\pi k^2 \sigma^2} \int_0^\infty q dq \int_0^{2\pi} d\varphi_{\mathbf{q}} \; e^{-(q/\sigma k)^2} e^{i\sqrt{k^2-q^2}z} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}$$

$$= \frac{2}{\pi\sigma^2} \int_0^\infty \xi d\xi \; e^{-(\xi/\sigma)^2} e^{i(kz)\sqrt{1-\xi^2}} J_0(k\rho\xi)$$

$$\psi_i(\mathbf{0}, 0) = 1$$

σ – scalar parameter (usually <<1)

$$k = \frac{\omega}{c}$$



Paraxial approximation:

$$kz \ll 4\pi/\sigma^4$$

$$\sqrt{1-\xi^2} \approx 1 - \frac{1}{2}\xi^2 - \frac{1}{8}\xi^4$$

$$\psi_i(\boldsymbol{\rho}, z) = \frac{1}{1 + i\sigma^2 kz/2} \exp\left[ikz - \frac{(\sigma k\rho/2)^2}{1 + i\sigma^2 kz/2}\right]$$

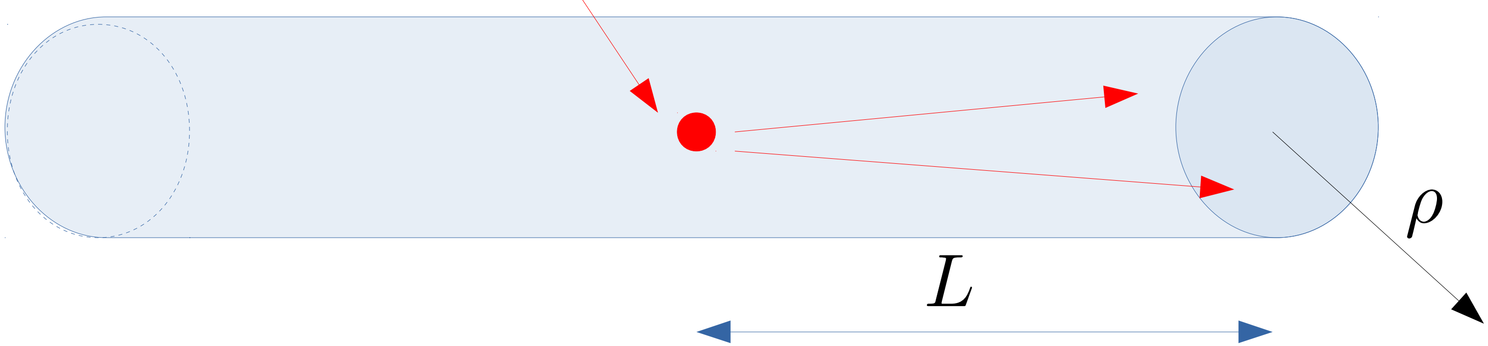
$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}, \quad w_0 = \frac{2}{\sigma k}, \quad z_0 = \frac{2}{\sigma^2 k}$$

Paraxial approximation (cont.)

So there exists a range of kz: $2/\sigma \ll kz \ll 2/\sigma^2$ in which:
(i) $z \gg w_0$ (propagation distance $\gg$ waist size)
(ii) paraxial approximation is accurate
(iii) there is no noticeable diffraction (pencil beam is a good approx.)

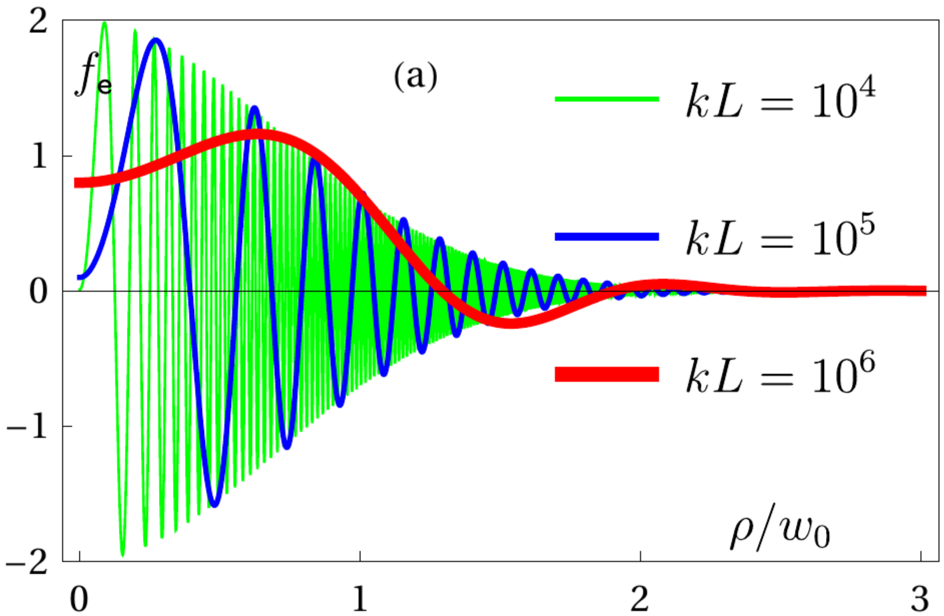
In simulations, we take $\sigma = 10^{-3}$ and $kL = 10^4,\ 10^5,\ 10^6$

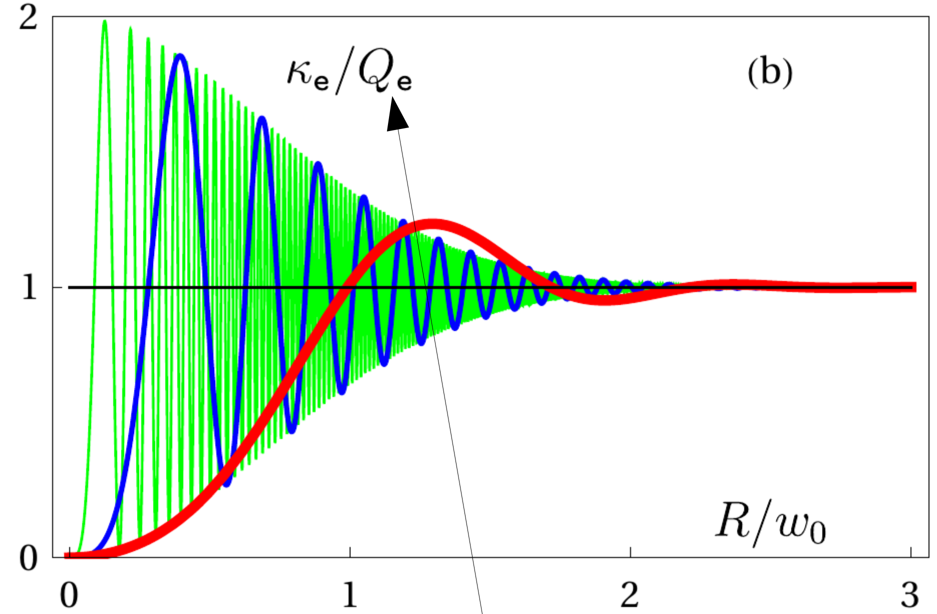
Small isotropically-scattering particle $\psi_s = \alpha \psi_i(0) \dfrac{e^{ikr}}{r}$



$$\kappa_e(R) = \int_{\mathbb{S}_- \cup \mathbb{S}_+} \mathbf{j}_e \cdot \hat{\mathbf{n}} d^2 r = \frac{2\pi}{L} \int_0^R f_e(\rho) \rho d\rho \ , \quad Q_e = \lim_{R \to \infty} \kappa_e(R)$$

Numerical evaluation of integrals

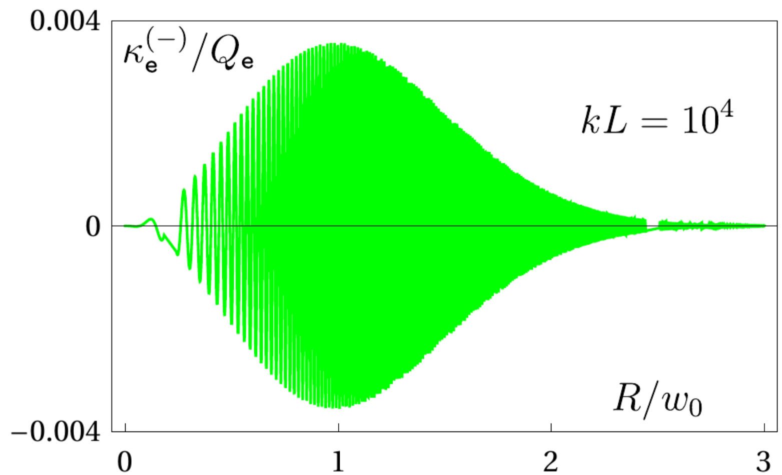




$$Q_e = 4\pi \mathrm{Im}\alpha$$

theoretical extinguished power (i.e., from optical theorem)

So, the “interaction area” stays approximately the same (a circle or radius w0) but oscillations of the integrand become slower, which exactly cancels the 1/*L* factor.



interference energy flux through the back-face z=-L

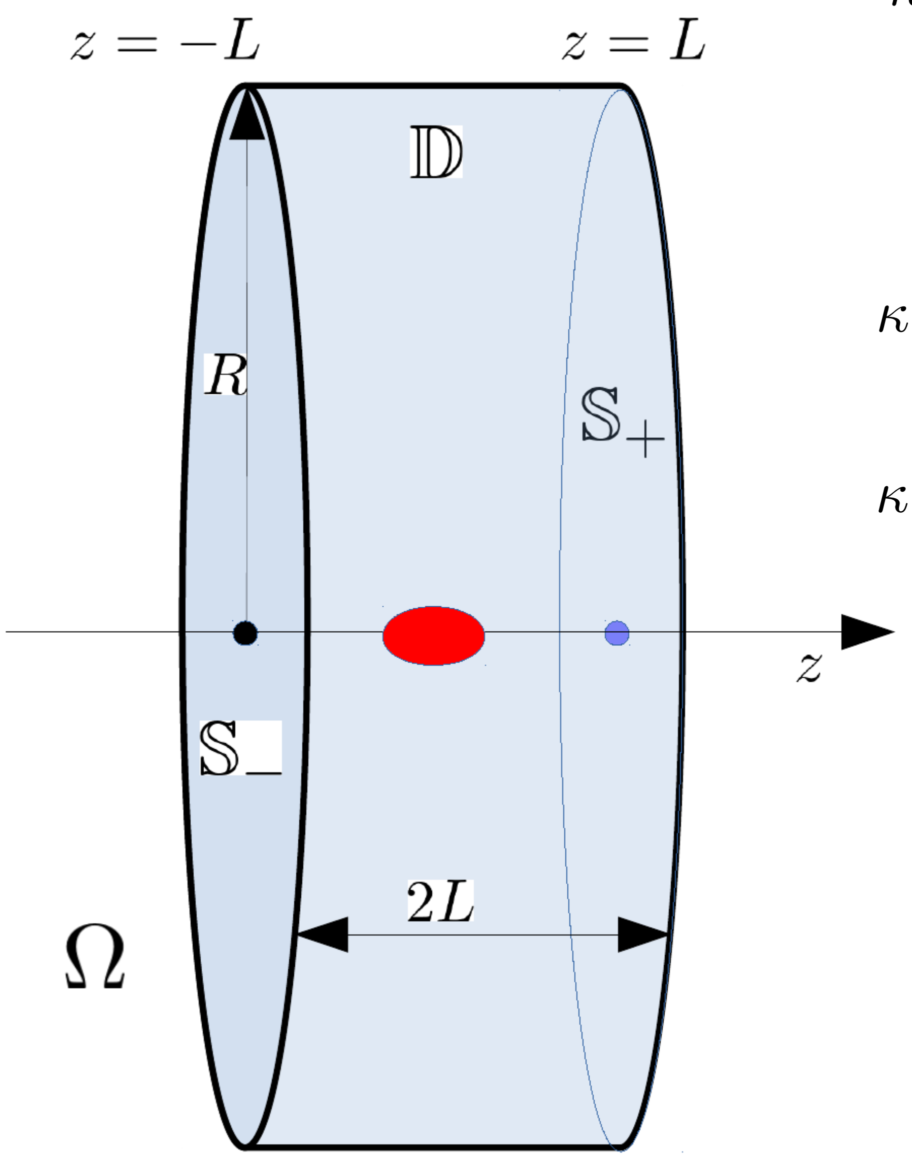
What about larger distances?

- We have considered the range $2/\sigma \ll kz \ll 2/\sigma^2$
- The paraxial approximation is valid for $kz \ll 4\pi/\sigma^4$
- The effect when the integrand becomes less oscillatory with *L* can not continue for ever.
- So, when $2/\sigma^2 \ll kz \ll 4\pi/\sigma^4$, the interaction spot starts to increase as *L*
- For even larger distances, the paraxial approximation breaks down. Computing the highly oscillatory integrals becomes difficult or impossible – but conservation of energy still works

Now we have complete resolution of the paradox:

- In the Fresnel diffraction region, the beam divergence is small or negligible, and the $1/L$ dependence of the scattered field is canceled by the oscillatory nature of the integral
- At larger distances (in the Fraunhoffer zone) the area of the interaction region starts to increase
- At even larger distances, the paraxial approximation breaks down and interference occurs

Incident plane wave



this is actually small, but is accounted for

$$\kappa_t(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot [\mathbf{j}_t(\rho,-L) - \mathbf{j}_t(\rho,L)]\rho d\rho$$

this quantity is measured with flat detectors

$$\kappa_s(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot [\mathbf{j}_s(\rho,-L) - \mathbf{j}_s(\rho,L)]\rho d\rho$$

$$\kappa_e(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot [\mathbf{j}_e(\rho,-L) - \mathbf{j}_e(\rho,L)]\rho d\rho$$

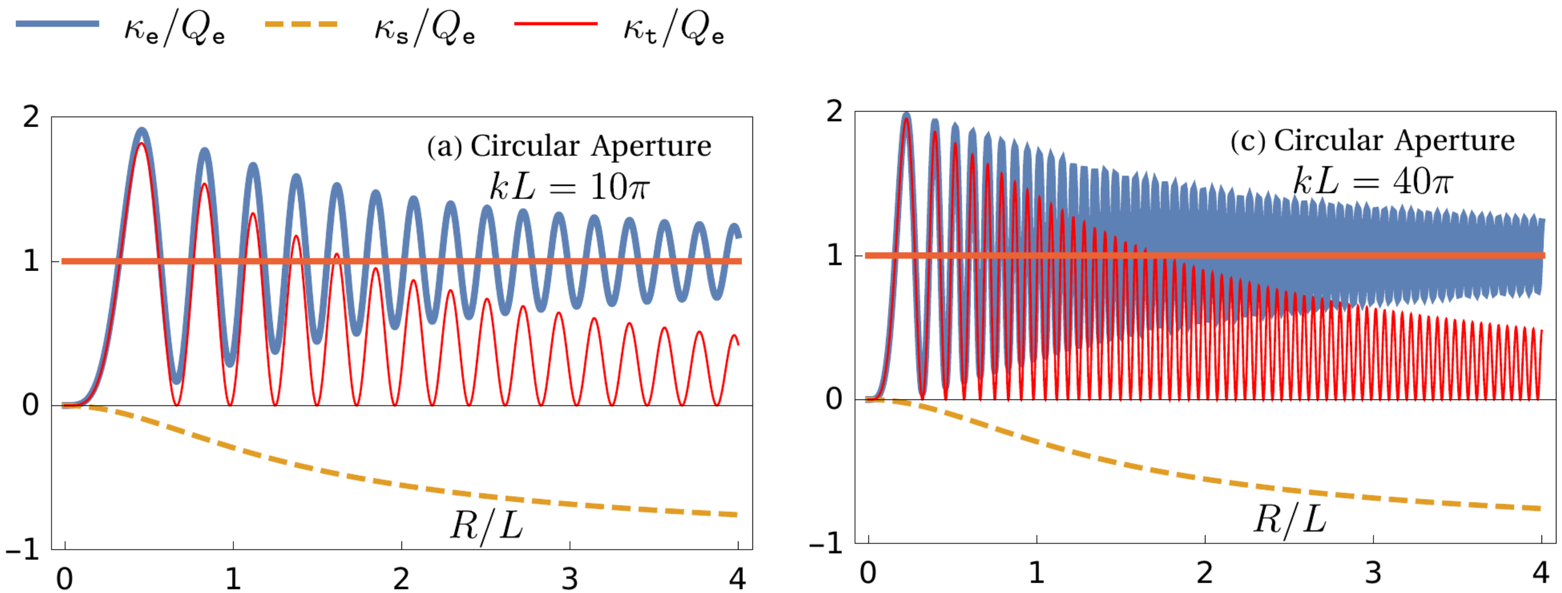
If sin(kL)=0 , we can expect that there is no flux through the cylindrical surface and:

$$\kappa_t(R) \xrightarrow[R\to\infty]{} Q_a$$

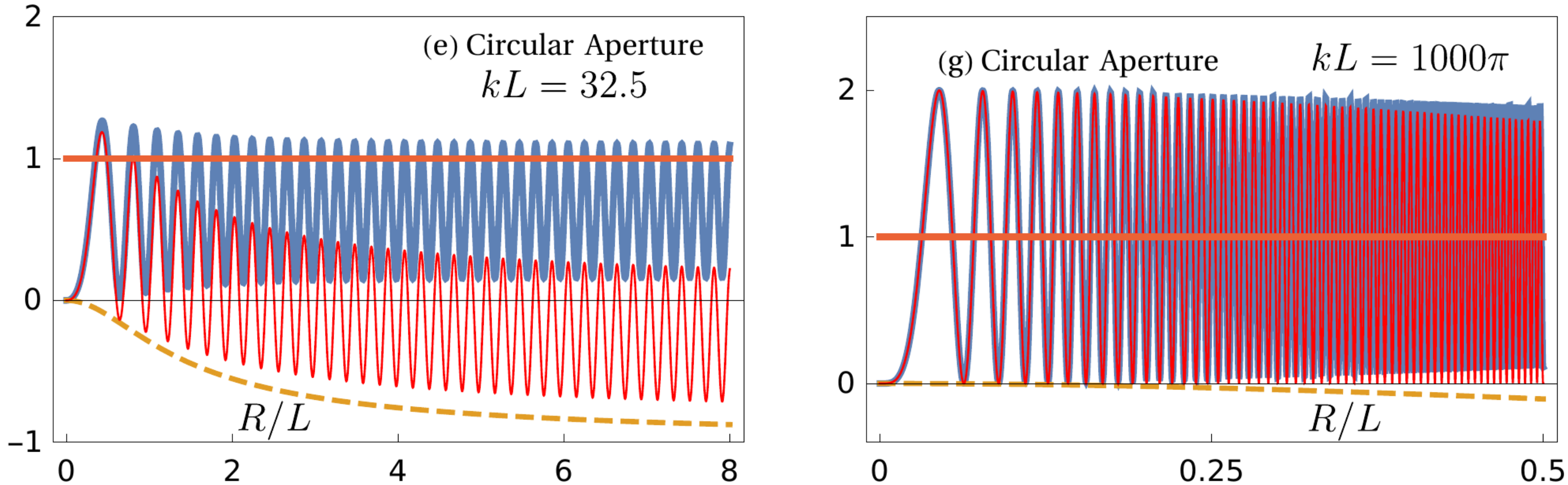
$$\kappa_s(R) \xrightarrow[R\to\infty]{} -Q_s$$

$$\kappa_e(R) \xrightarrow[R\to\infty]{} ?$$

a) Circular aperture nonabsorbing small particle (Qa=0, Qs=Qe)



In this plot sin(*kL*)=/=0



a) Circular aperture (cont.)

The above plots can be easily understood because the fluxes can be all computed analytically:

$$\kappa_e(R) = 4\pi[I_1(R)\cos(kL) + I_2(R)\sin(kL)]$$

$$I_1(R) = \mathrm{Im}\left[\alpha\left(e^{ikL} - \frac{Le^{ik\sqrt{L^2+R^2}}}{\sqrt{L^2+R^2}}\right)\right]$$

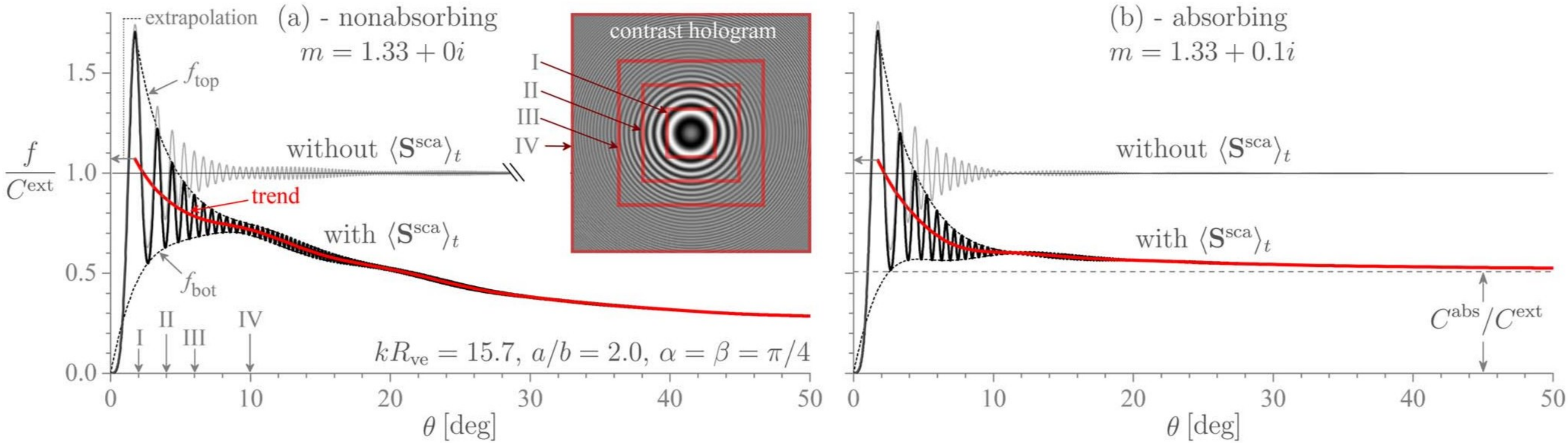
$$I_2(R) = \mathrm{Im}\left[\alpha\left(e^{ik\sqrt{L^2+R^2}} - e^{ikL}\right)\right]$$

$$\Downarrow \quad e^{ikL} = \pm 1$$

$$\kappa_e(R) = 4\pi\mathrm{Im}\left[\alpha\left(1 \pm \frac{Le^{ik\sqrt{L^2+R^2}}}{\sqrt{L^2+R^2}}\right)\right] \xrightarrow[R\to\infty]{} Q_e \equiv 4\pi\mathrm{Im}\alpha$$

a) Circular aperture: how can we use flat detectors to measure extinction?

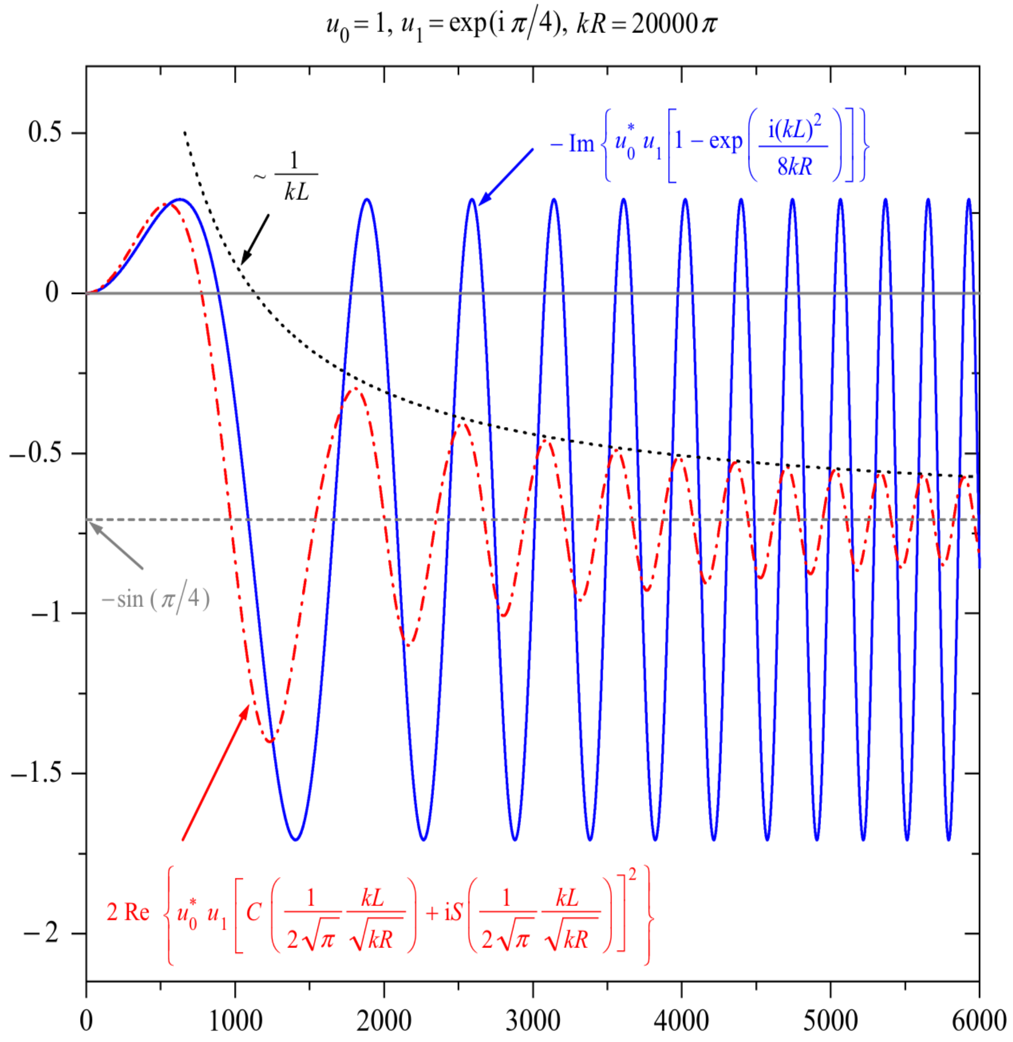
Try to interpolate the oscillatory behavior

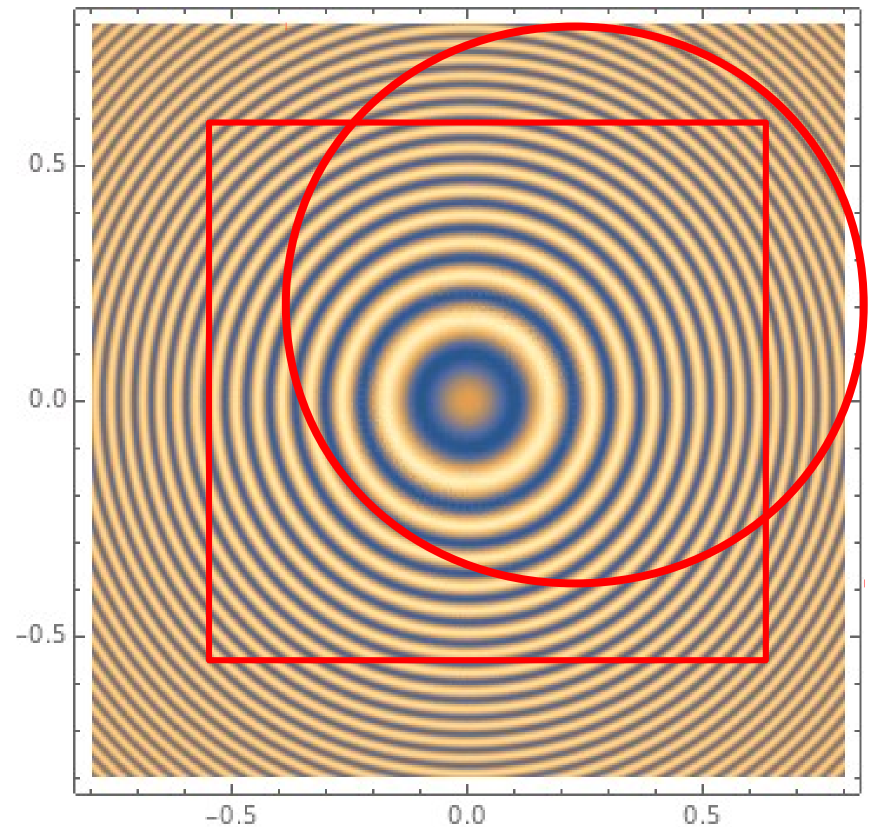


From M.J.Berg, N.R. Subedi and P.A.Anderson, Optics Letters 42, 1011, 2017

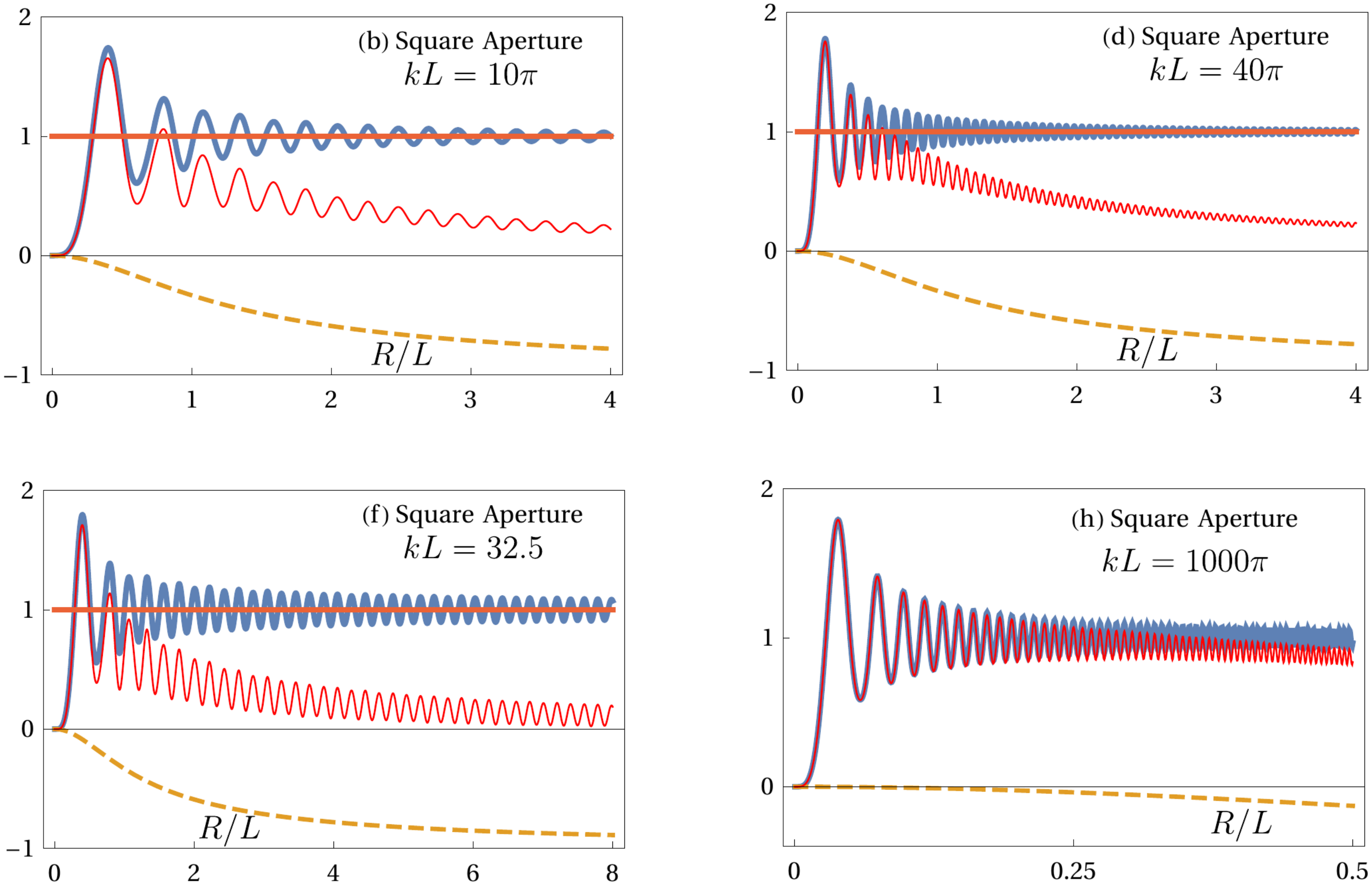
b) Or try a non-circular (square) aperture in the far Fraunhofer zone

The critical insight was made in M.I.Mishchenko, M.J. Berg, C.M. Sorensen and C.V.M. van der Mee, J.Quant.Spect.Rad.Trans. 110, 323, 2009

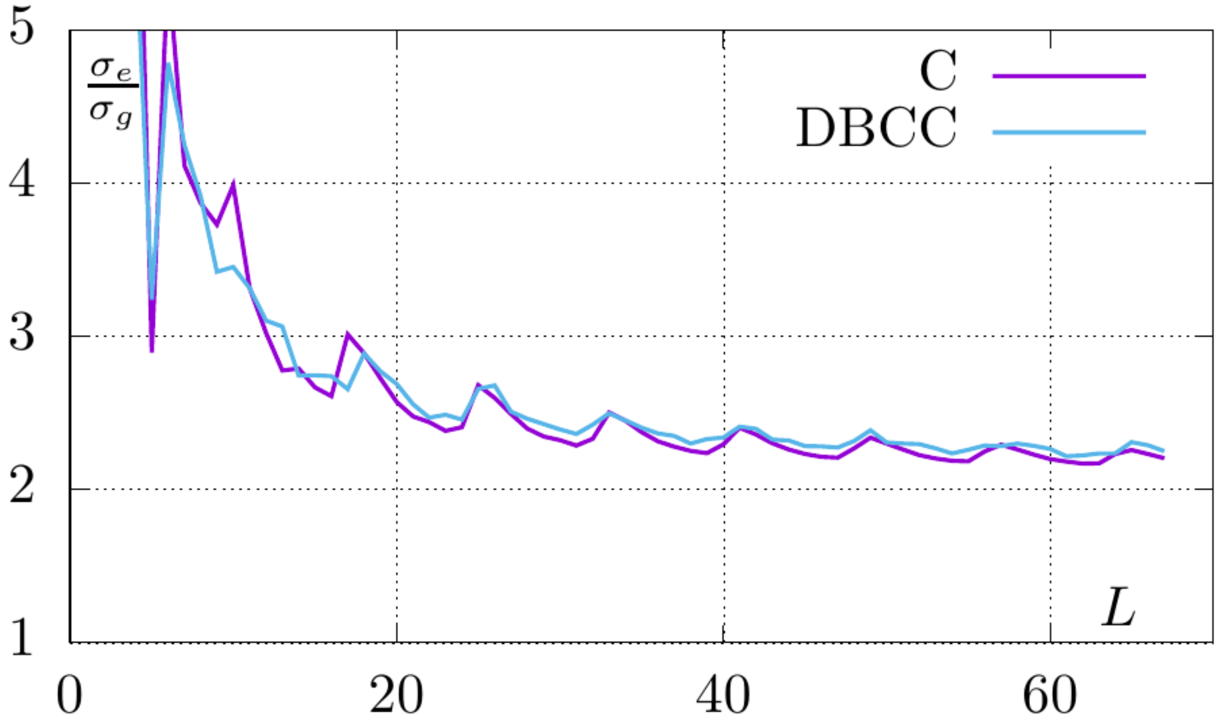




b) Square aperture, far Fraunhofer zone



Paradox 2: (Classical)



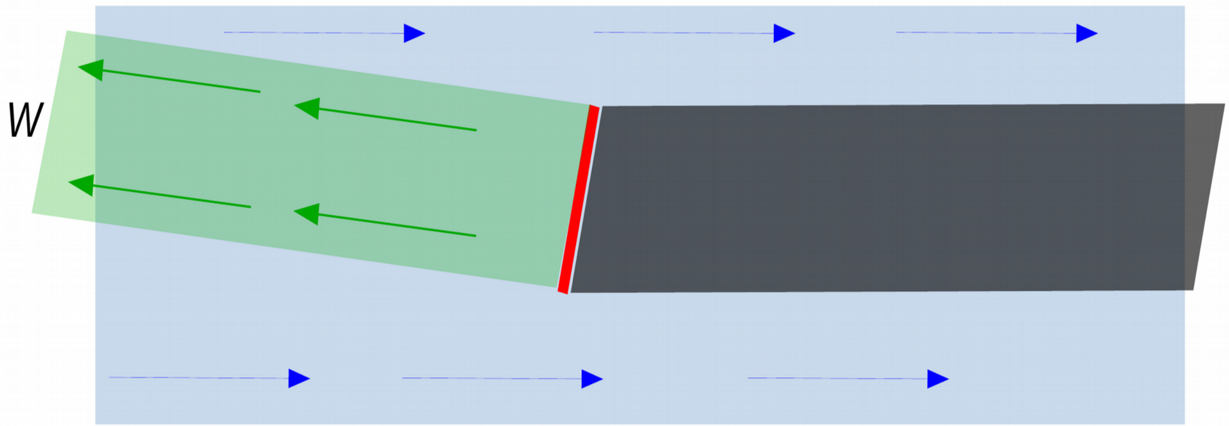
Usually formulated for spheres and Mie scattering but here it is demonstrated for a metal parallelepiped slightly below the plasma frequency.

The extinction cross section is roughly two times larger than the geometrical cross section.

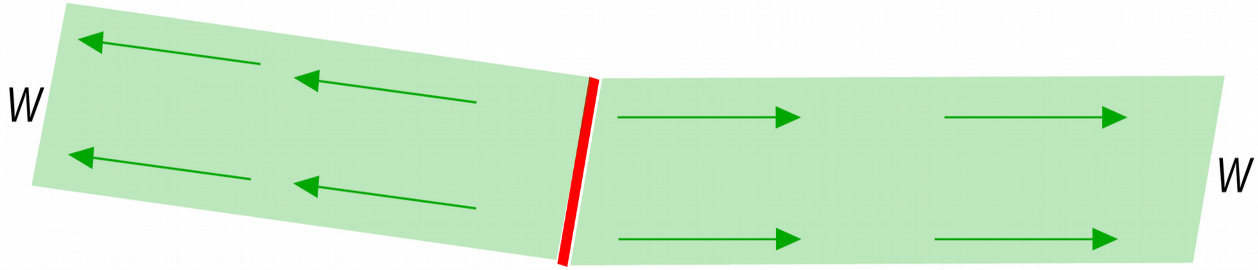
Incorrect explanation: edge diffraction (still common today)

Conceptually-correct but somewhat complicated explanations:
L. Brillouin, J. Appl. Phys. 20, 1110 (1949)
D. M. LeVine, NASA Technical report (1983)
H. M. Lai, W. Y. Wong, W. H. Wong, JOSAA 21, 2324 (2004) – 2D
M. J. Berg, C. M. Sorensen, A. Chakrabarti, JQSRT 112, 1170 (2011)
– occurs for transparent materials as well!

Geometrical optics explanation of the classical extinction paradox



(a) An incident energy flux is blocked by a slightly tilted mirror.

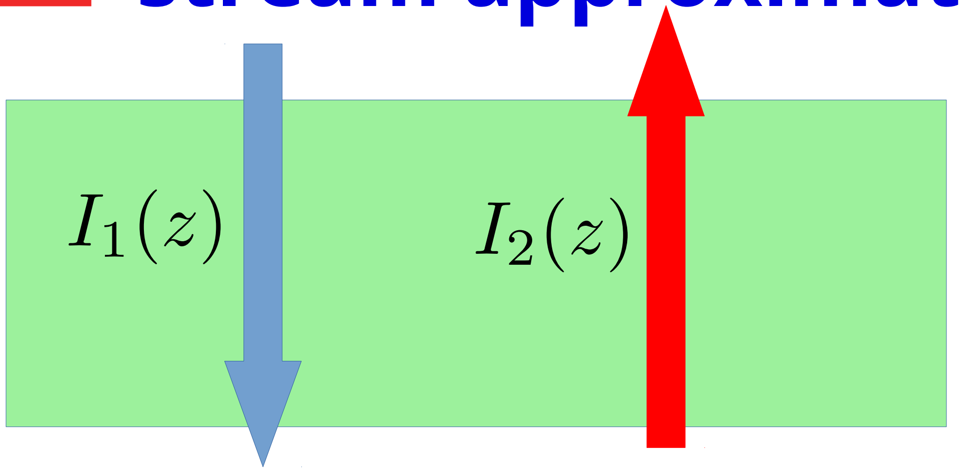


(b) The light actually scattered by the mirror. The right-going flux cancels the incident flux to create the geometrical shadow shown in Panel (a).

Many particles

- No multiple scattering – previous analysis largely applies. Each particle remove the power Qe from the incident plane wave. We can use statistical analysis to derive the rate at which the energy reaching the detector decays.
- Multiple scattering is significant – Extinction cross section of a single particle is insufficient to predict energy reaching the detector. We also need at least the phase function.

Phase function (simple example for two-stream approximation)



[P. Kubelka and F. Munk, Z.Tech. Phys.12, 593 (1931)]

number of particles in a unit volume

Average absorption cross section

$$\mu_a = \rho\sigma_a$$

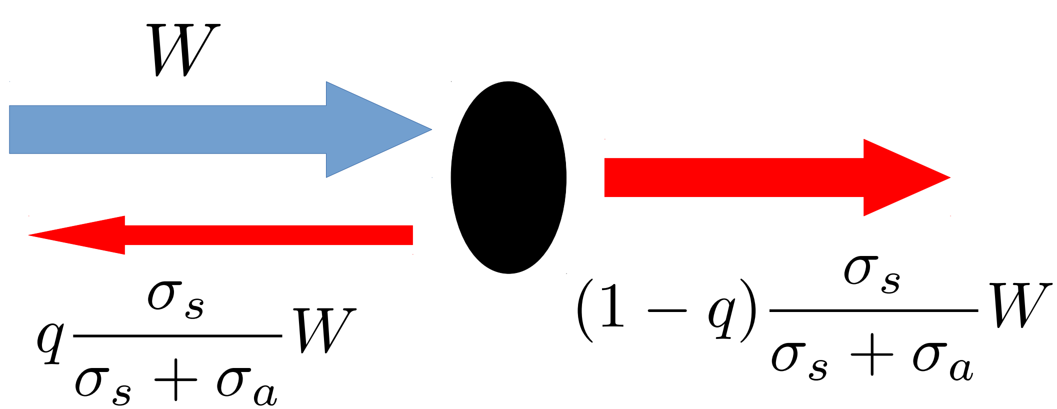
$$\mu_s = \rho\sigma_s$$

$$(d/dz + \mu_a + q\mu_s)I_1(z) = q\mu_s I_2(z)$$

$$(-d/dz + \mu_a + q\mu_s)I_2(z) = q\mu_s I_1(z)$$

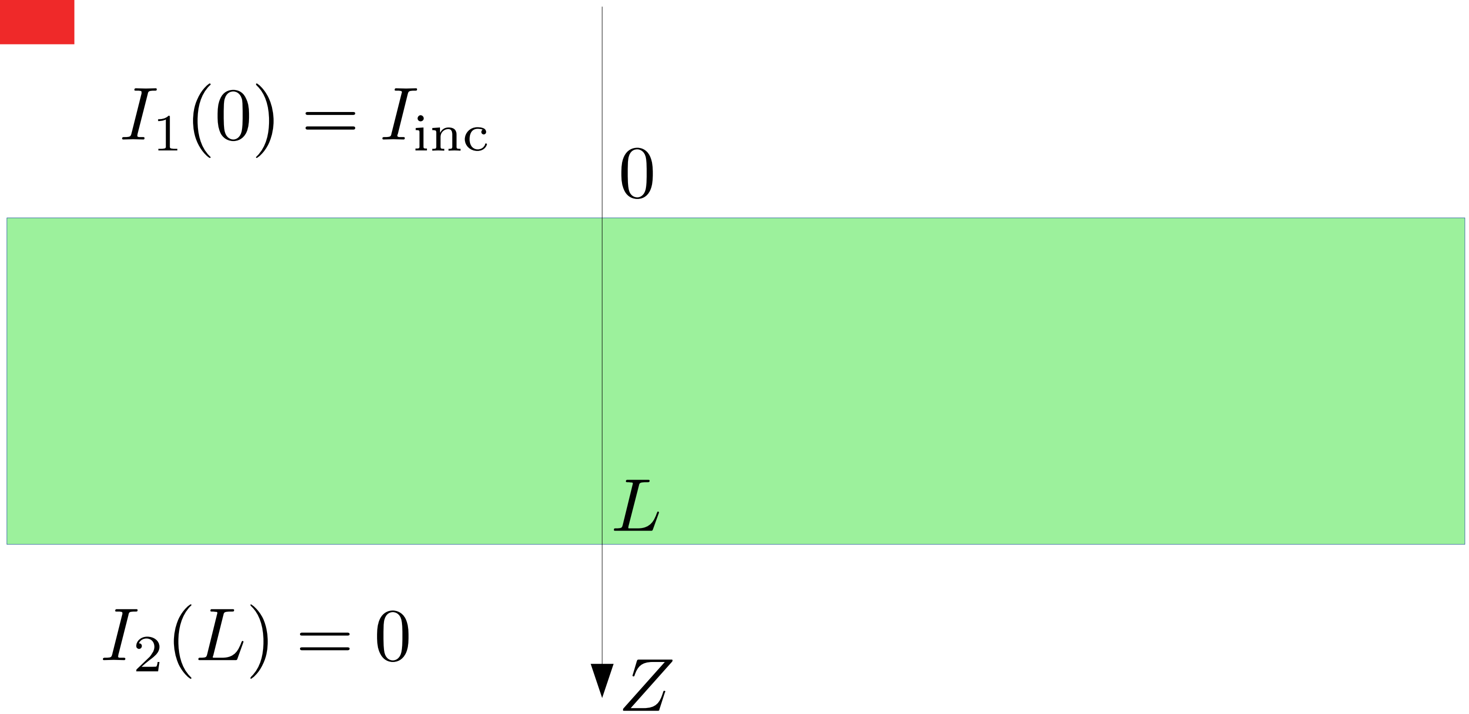
probability (or fraction of energy) that is scattered backwards

$$q < 1$$



Two-stream solutions

$$I_1(0) = I_{\text{inc}}$$



$$I_2(L) = 0$$

$$I_1(z) = \frac{a_- e^{\lambda(z-L)} - a_+ e^{\lambda(L-z)}}{a_- e^{-\lambda L} - a_+ e^{\lambda L}}$$

$$I_2(z) = q\mu_s \frac{e^{\lambda(z-L)} - e^{\lambda(L-z)}}{a_- e^{-\lambda L} - a_+ e^{\lambda L}}$$

$$a_\pm = \mu_a + q\mu_s \pm \lambda$$

$$\lambda = \sqrt{\mu_a(\mu_a + 2q\mu_s)}$$

This is the exponent that defines the rate of decay ("diffuse wavelength")

Transmission and reflection in the two-stream model

$$T = \frac{\lambda}{(\mu_a + q\mu_s)\sinh(\lambda L) + \lambda\cosh(\lambda L)}$$

$$R = \frac{q\mu_s \sinh(\lambda L)}{(\mu_a + q\mu_s)\sinh(\lambda L) + \lambda\cosh(\lambda L)}$$

a) Non-absorbing layer:

$$T = \frac{1}{1 + q\mu_s L}\ , \quad R = \frac{q\mu_s L}{1 + q\mu_s L} \qquad (\text{if } \mu_a \to +0)$$

$$q\mu_s L \ll 1 \quad : \ \text{white clouds (viewed from ground)}$$

$$q\mu_s L \sim 1 \quad : \ \text{grey clouds}$$

$$q\mu_s L \gg 1 \quad : \ \text{black clouds}$$

b) Optically-thick layer with finite absorption:

$$T \xrightarrow[L \to \infty]{} 0$$

$$R \xrightarrow[L \to \infty]{} R_\infty = \frac{q\mu_s}{\mu_a + q\mu_s + \lambda}$$

(can be close to 1)

Partially-coherent and non-monochromatic light

- Coherence theory
- Fluctuation-dissipation theorem
- Optical theorem can be generalized to partially-coherent fields

  P.S.Carney, E.Wolf, G.S.Agarwal, JOSAA 14, 3366 (1997)
- All these theories concern averages
- Fluctuations, especially in systems with optical resonances, is an open question

Local definitions of extinction and absorption

- In a stationary situation, absorbed power is the total work done by the total field (whatever exists in the medium) per unit time.
- The extinguished power is the total work done by the incident field (on the actual currents)
- Optical theorem can be derived from this statement for an incident plane wave

For a monochromatic, plane **electromagnetic** wave of amplitude E_0:

$$\sigma_a = \frac{(4\pi)^2 k \mathrm{Im}\epsilon}{|\mathbf{E}_0|^2 |\epsilon - 1|^2} \int |\mathbf{P}(\mathbf{r})|^2 d^3 r$$

$$\mathbf{P}(\mathbf{r}) = \frac{\epsilon(\mathbf{r}) - 1}{4\pi} \mathbf{E}(\mathbf{r})$$

$$\sigma_e = \frac{4\pi k}{|\mathbf{E}_0|^2} \int \mathbf{E}^*_{\mathrm{inc}}(\mathbf{r}) \cdot \mathbf{P}(\mathbf{r}) d^3 r$$

Conclusions

- Extinction is a very robust property of particles to remove power from an incident beam or wide-front radiation
- However, some physical situations involving extinction are surprisingly complex
- All paradoxes can be resolved by working from first principles
- In the case of multiple scattering, extinction cross section of a particle is not sufficient forn a complete complete characterization of the medium.