TOMOGRAPHIC IMAGING OF SCATTERING MEDIA

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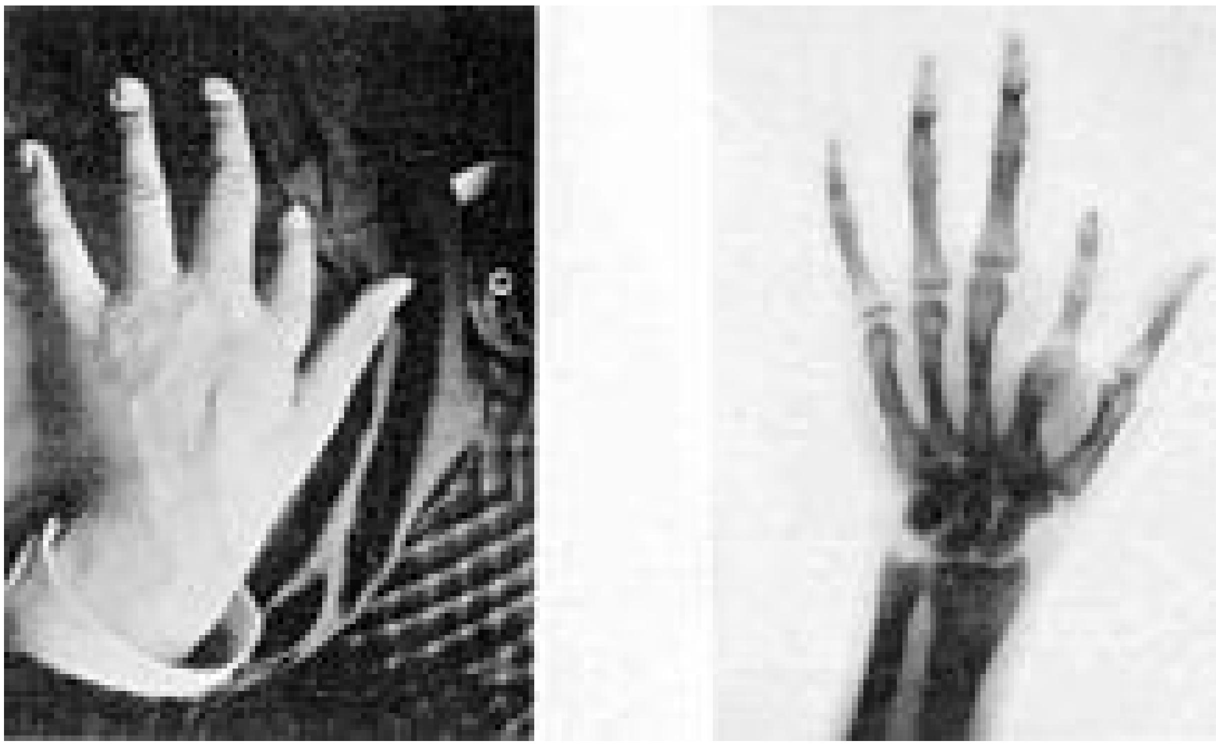
	X-ray tomography, PET, SPECT	**Diffraction tomography , near-field tomography, acoustic imaging**	**Optical diffusion tomography (mostly, IR)**
Diffusion equation $\alpha(\mathbf{r}) = c\mu_a(\mathbf{r})$ $D(\mathbf{r}) = \frac{c}{3\left[\mu_a(\mathbf{r}) + (1-g)\mu_s(\mathbf{r})\right]}$	In principle, applicable, but not in realistic samples	N/A	YES (The most frequent approach)
Radiative transport equation $\mu_a(\mathbf{r}),\ \mu_s(\mathbf{r}),\ A(\hat{\mathbf{s}},\hat{\mathbf{s}}')$ $\mu_t(\mathbf{r}) = \mu_a(\mathbf{r}) + \mu_s(\mathbf{r})$	YES	May be, if geometrical optics or a similar approx. is used	YES
Macroscopic Maxwell's Equations or some form of wave equation Permittivity $\varepsilon(\mathbf{r})$ Index of refraction $n(\mathbf{r})$	N/A	YES	In principle applicable but too detailed, intractable
MICROSCOPIC THEORY (Quantum mechanics, condensed matter theory…)			

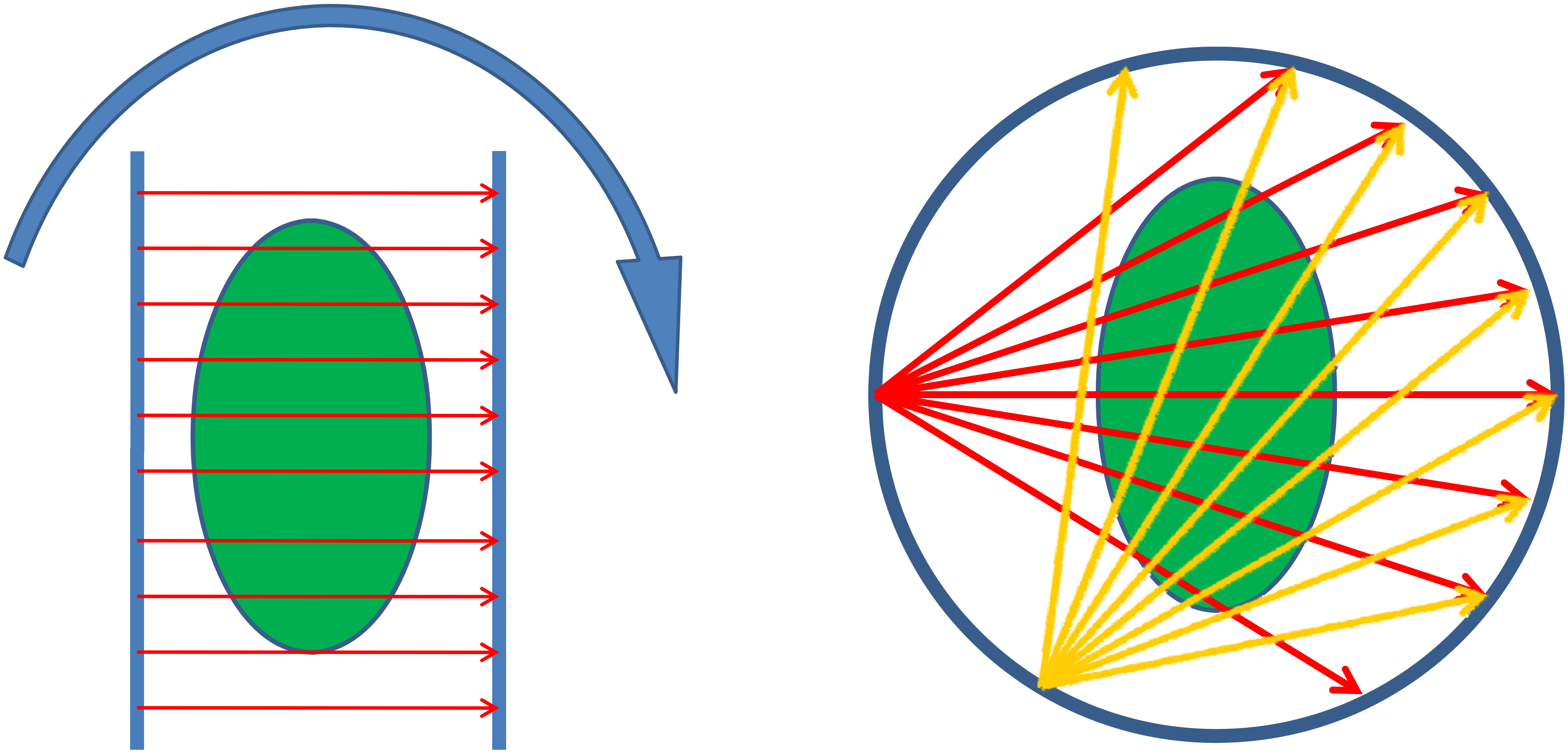
OUTLINE: PHYSICAL REGIMES OF SCATTERING

1. Weak scattering: Single-scattering tomography and broken ray transform (BRT)
2. Strong scattering regime: Optical diffusion tomography (ODT)
3. Intermediate scattering regime: Inverting the radiative transport equation (RTE)
4. Nonlinear problem of inverse scattering

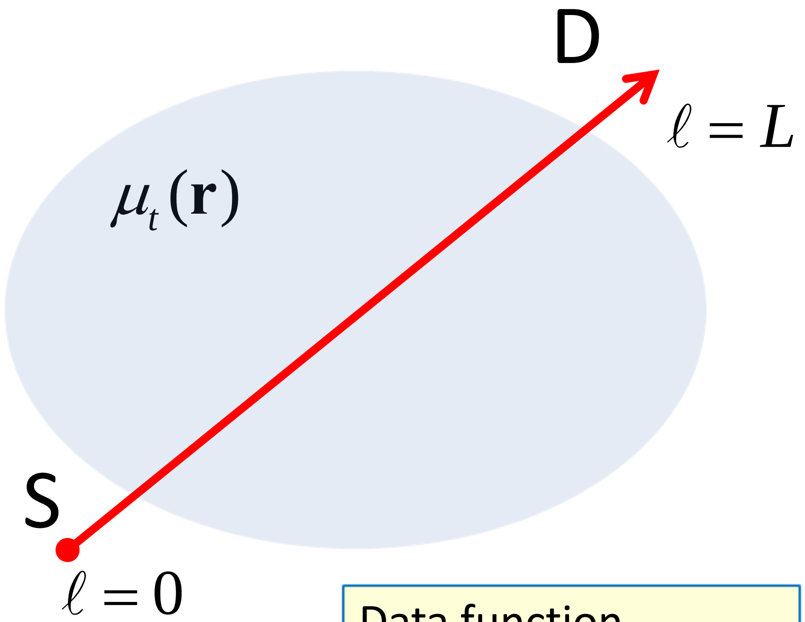
1. WEAK SCATTERING REGIME



Conventional X-Ray Tomography Disregards Scattering



If we can associate a well-define trajectory with every source-detector pair, the inverse problem is usually linear



$$I_D = I_S \exp\left[-\int_0^L \mu_t(\ell)\, d\ell\right]$$

$$\phi(S,D) \equiv -\ln\left(\frac{I_D}{I_S}\right)$$

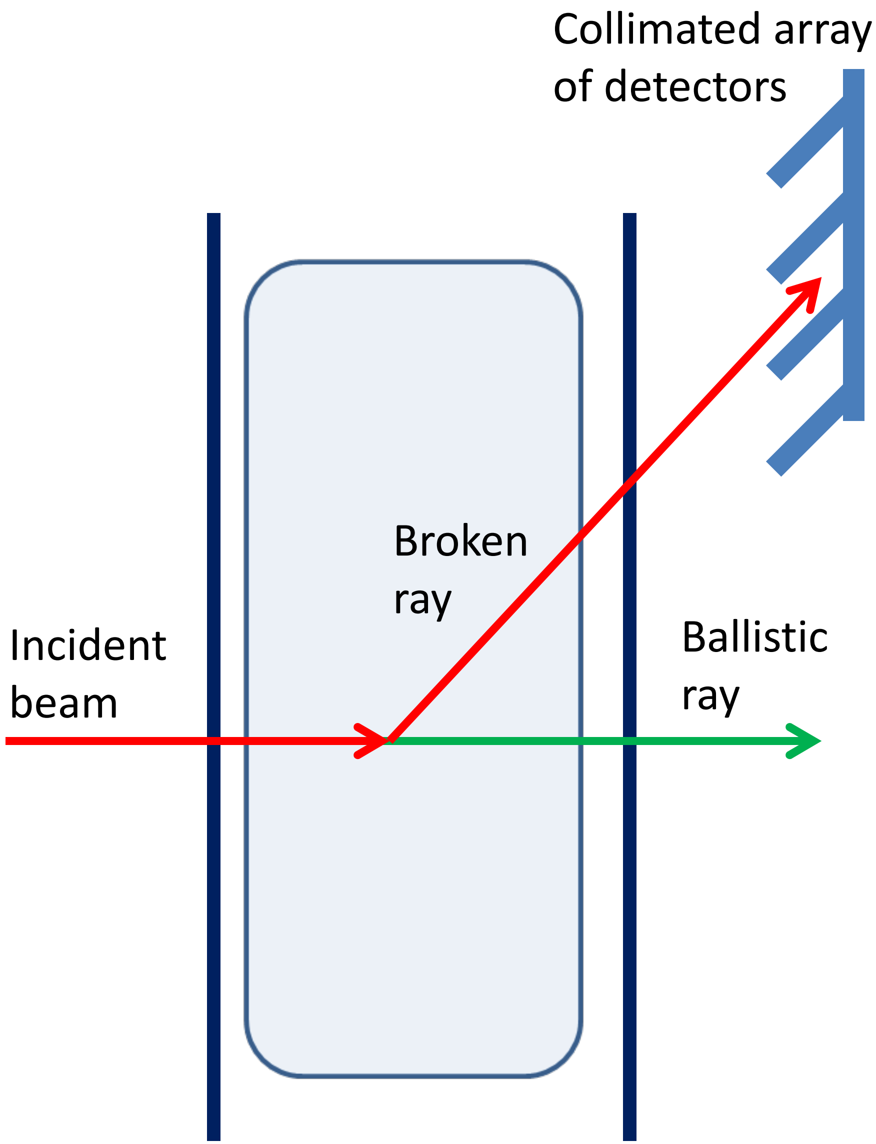
$$\phi(S,D) = \int_0^L \mu_t(\ell)\, d\ell$$

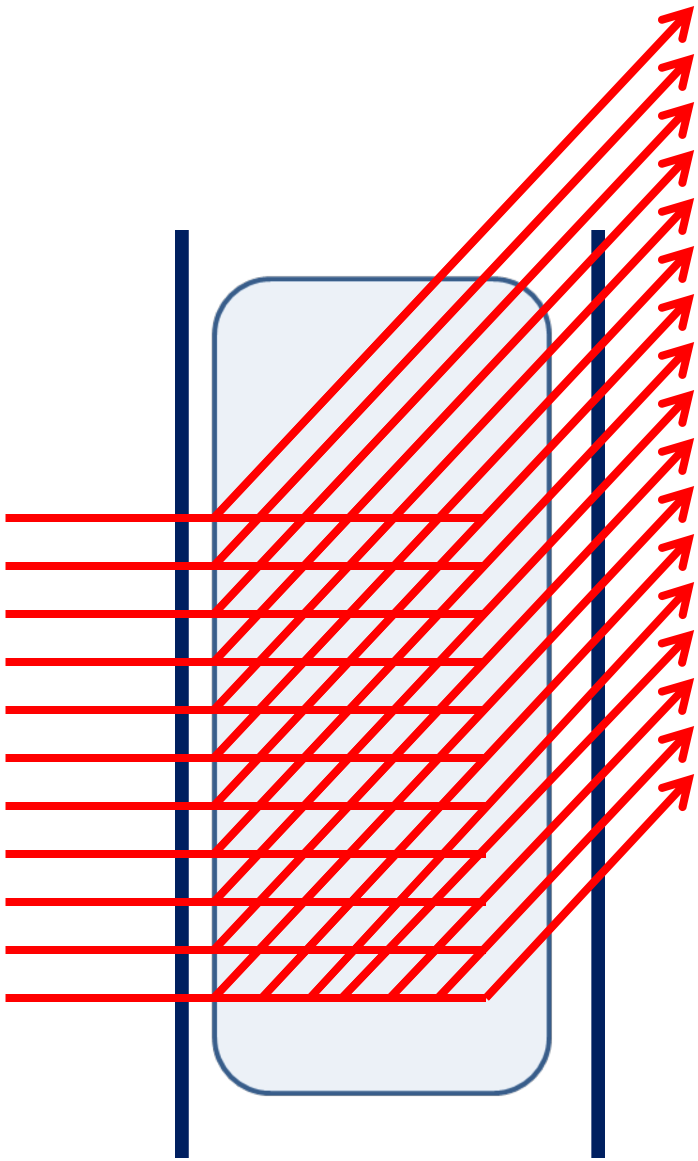
(the Radon transform)

Data function (defined for each source-detector pair)

The unknown function (to be reconstructed)

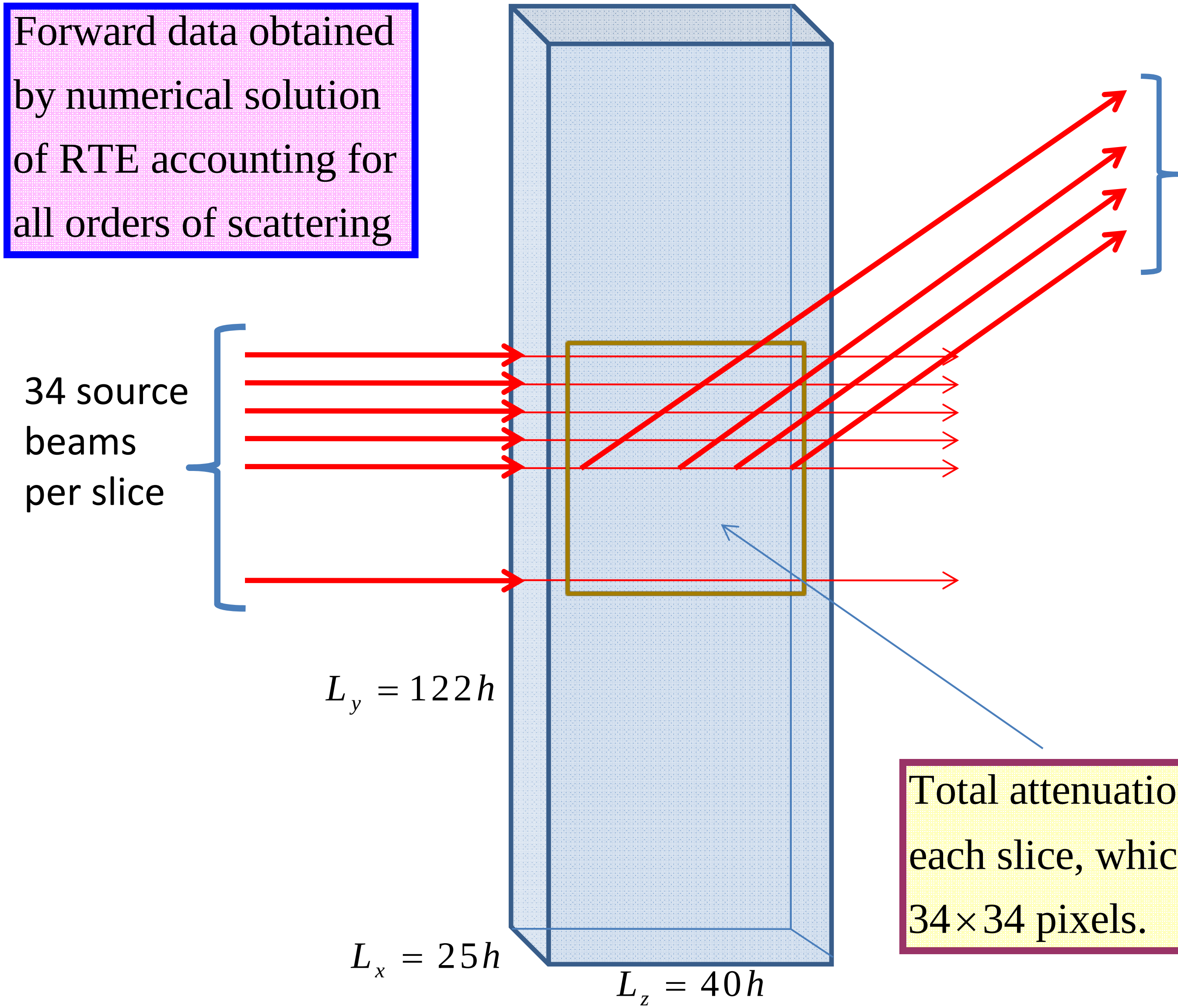
What if we take first-order scattering into account?





Initial Numerical Experiment for Constant Scattering

Forward data obtained by numerical solution of RTE accounting for all orders of scattering



const

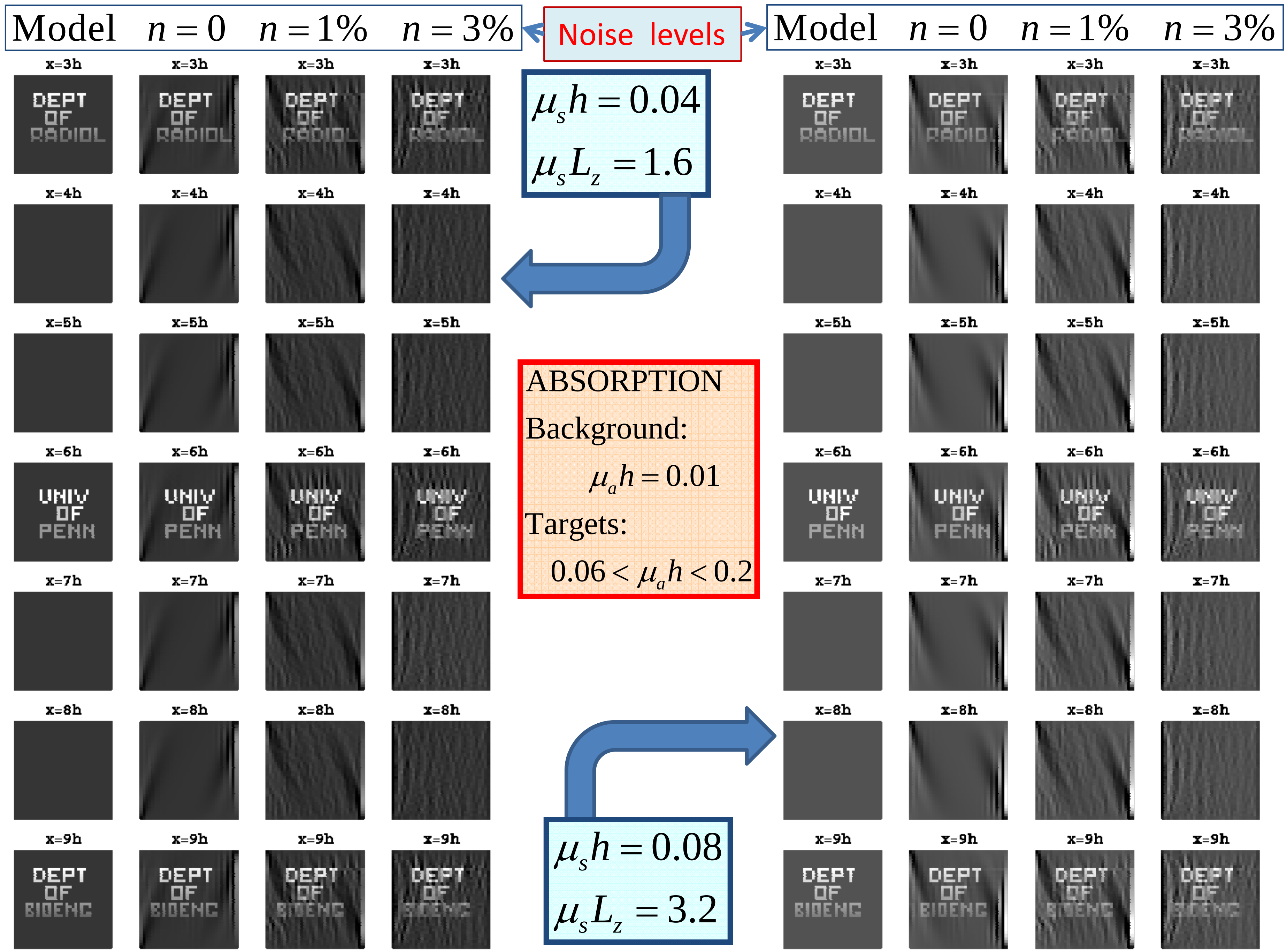
$$\mu_t(\mathbf{r}) = \mu_s + \mu_a(\mathbf{r})$$

(a) $\mu_s L_z = 1.6$

(b) $\mu_s L_z = 3.2$

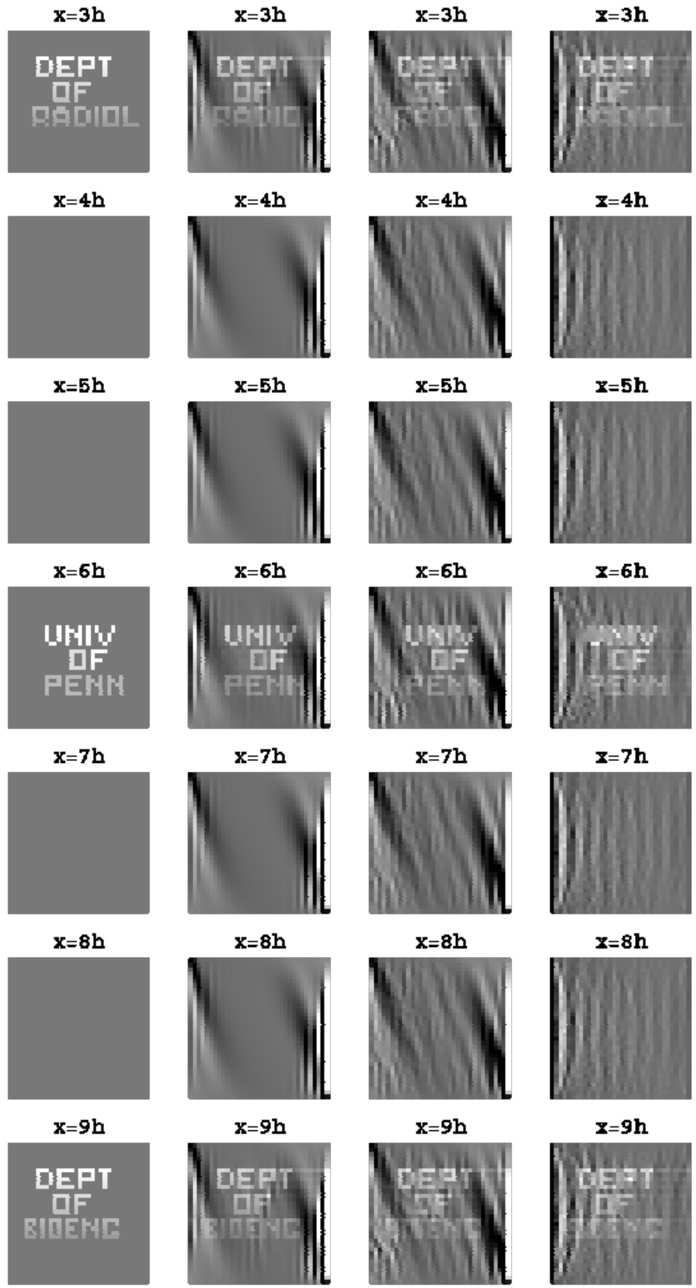
(c) $\mu_s L_z = 6.4$

Total attenuation μ_t is reconstructed in each slice, which is rasterized using 34×34 pixels.



Model $n=0$ $n=1\%$ $n=3\%$
Noise levels
Model $n=0$ $n=1\%$ $n=3\%$
$\mu_s h = 0.04$
$\mu_s L_z = 1.6$
ABSORPTION
Background:
$\mu_a h = 0.01$
Targets:
$0.06 < \mu_a h < 0.2$
$\mu_s h = 0.08$
$\mu_s L_z = 3.2$
x=3h
x=4h
x=5h
x=6h
x=7h
x=8h
x=9h
DEPT OF RADIOL
UNIV OF PENN
DEPT OF BIOENG

Model $n=0$ $n=1\%$ $n=3\%$



$$\mu_s = 0.16h^{-1}$$

$$\mu_s L_z = 6.4$$

L.Florescu, J.C.Schotland and V.A.Markel ,
Phys. Rev. E 79, 036607, 2009.

A few useful features:

- The inverse problem is linear and mildly ill-posed
- Method works up to surprisingly large optical depths
- No multiple projections are required!

However….

- We have performed so far only a purely numerical reconstruction for a small grid of 34x34 pixels per slice. In practical applications, we want much larger grids.
- We have assumed that scattering is spatially uniform. But this is an unrealistic assumption!
- Can we derive fast and stable image reconstruction formulas (similar to filtered backprojection formula) that would work for spatially-nonuniform scattering coefficient?

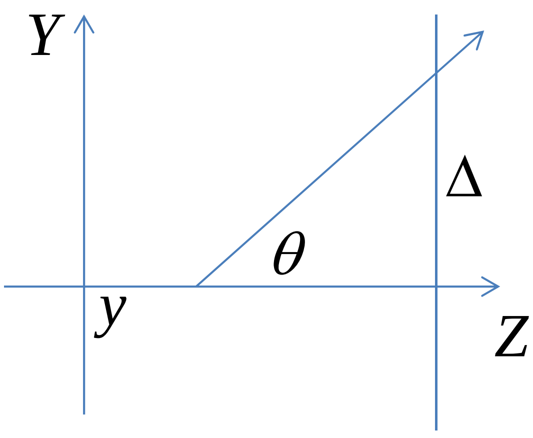
For constant scattering, there exists an analytical image reconstruction formula, which is similar to the filtered backprojection formula ...

(do not pay much attention to the math details)

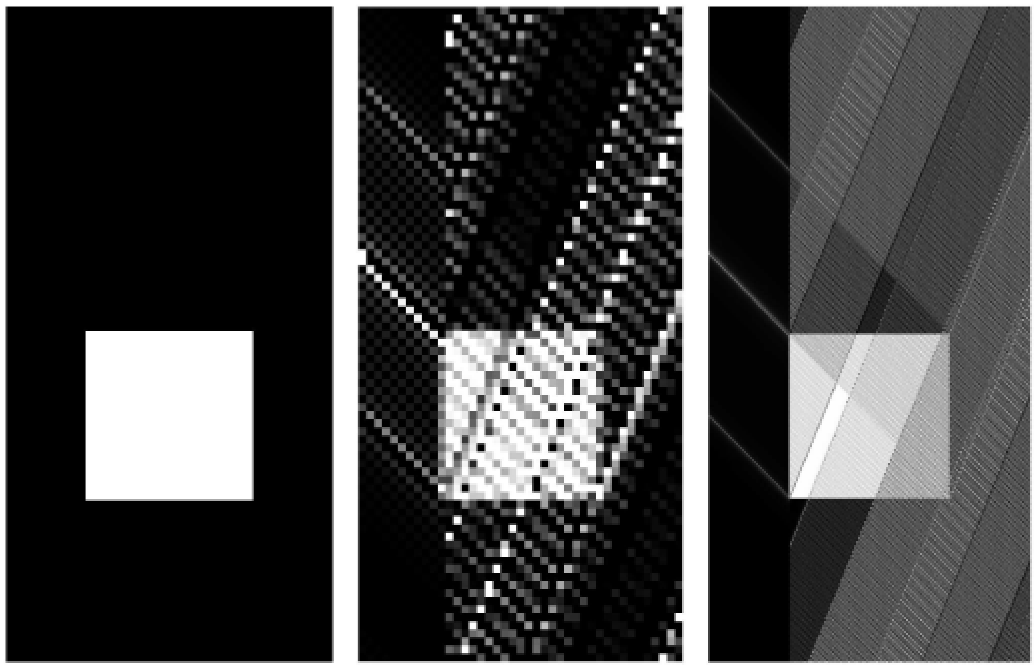
$$\mu_t(y,z) = \sigma \left\{ \left(\frac{\partial}{\partial \Delta} - \frac{\partial}{\partial y} \right) \phi(y,\Delta) + \frac{\sigma}{\tau} \frac{\partial}{\partial y} \left[\phi(y+\sigma z, \tau L) - \phi(y,\Delta) \right] \right.$$
$$\left. \left. - \left(1 + \frac{\sigma}{\tau} \right) \frac{\partial}{\partial y} \int_{\Delta}^{\tau L} \phi\left(y + \frac{\sigma}{\tau}(\ell - \Delta), \ell \right) d\ell \right\} \right|_{\Delta = \tau(L-z)}$$

$$\sigma = \mathrm{ctg}(\theta/2) \; ; \; \tau = \mathrm{tg}\,\theta$$

... but it is unstable in the presence of sharp discontinuities of the target.

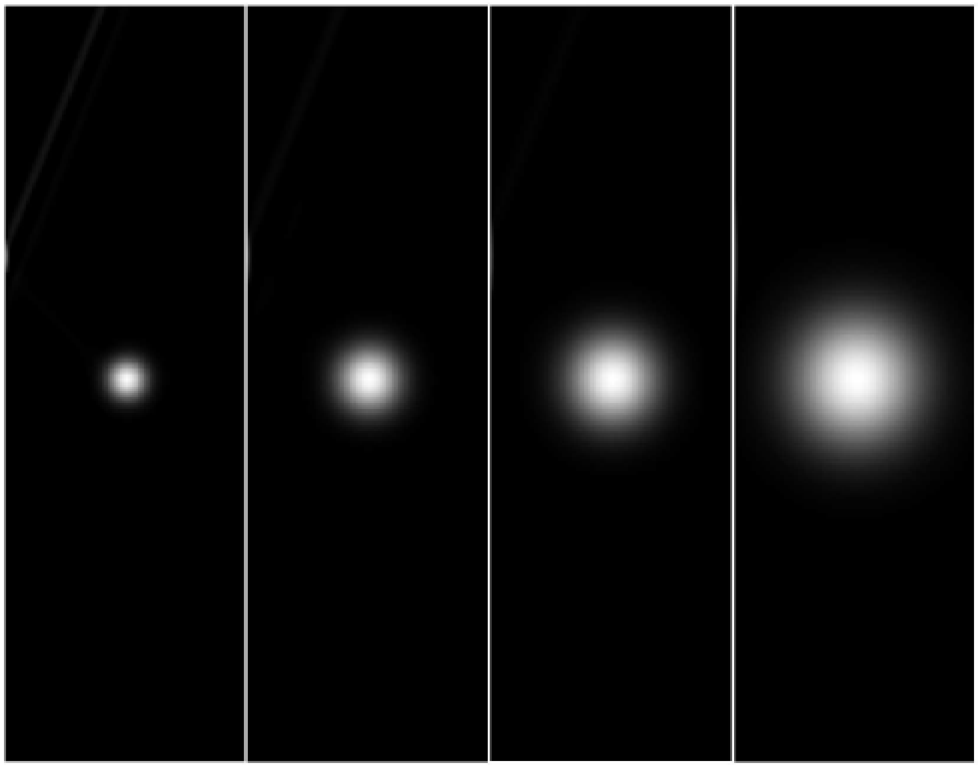


Reconstructions of an absorbing square using the analytical formula



Model L/h=40 L/h=400

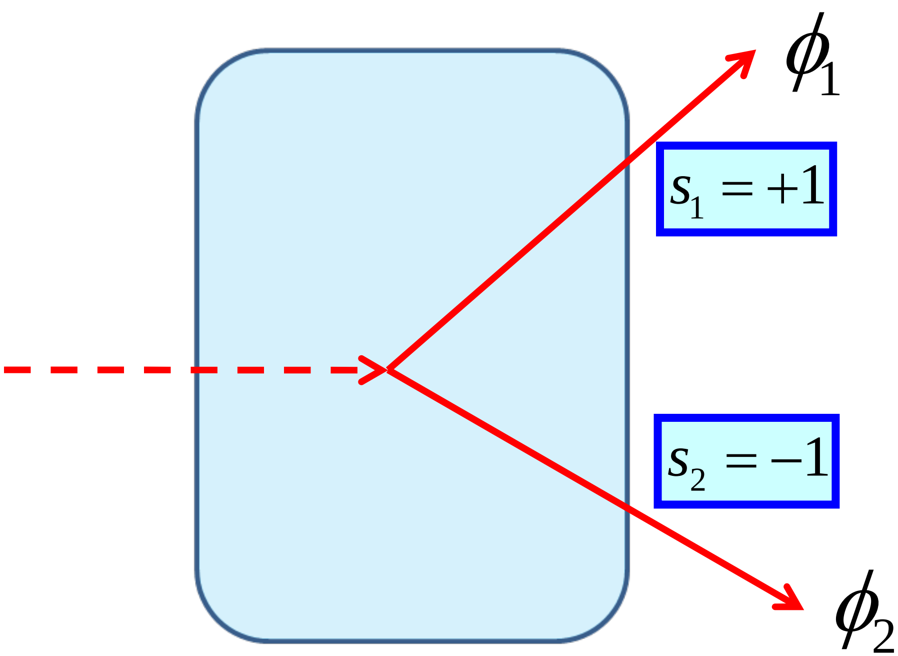
... works OK for Gaussians, but this is not a physically interesting case.



$$\frac{w}{L} = \frac{3}{40} \qquad \frac{w}{L} = \frac{5}{40} \qquad \frac{w}{L} = \frac{7}{40} \qquad \frac{w}{L} = \frac{10}{40}$$

L.Florescu, V.A.Markel and J.C.Schotland ,
Inverse Problems **27**, 025002, 2011.

What about spatially non-uniform scattering? Need to use at least two broken rays per one incident ray and define a linear combination of measurements.

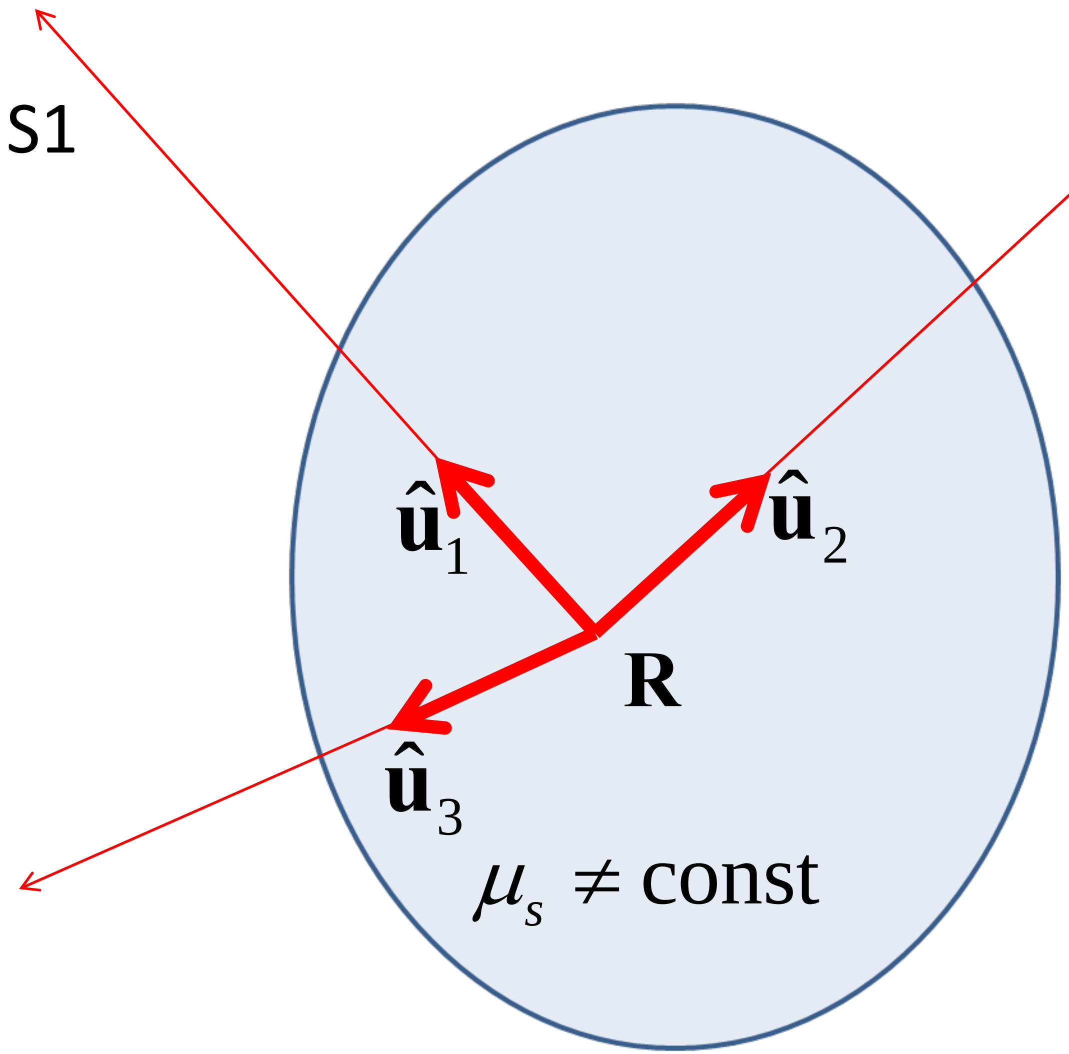


$$\phi = \phi_1 - \phi_2$$

(i) Also ill-posed and does not work very well (examples are not shown)

(ii) Note that the ray shown by the dashed line is canceled from the integral transform

Generalization of the broken-ray transform: *K*-star transform (shown below for the special case *K*=3)



With the source S1 on the detector D2 would measure the d.f. $\phi_{12}(\mathbf{R})$

The star transform is constructed as a linear combination of such functions

Mathematical Formulation of the Star Transform

$$\Phi\left(\mathbf{R}\right) = \sum_{k=1}^{K} s_k I_k\left(\mathbf{R}\right) \quad ,$$

$$\mathbf{R} \equiv (Y, Z) \in \overline{S} = \{0 \le Z \le L\} \quad ,$$

$$I_k\left(\mathbf{R}\right) = \int_0^{\ell_k(Z)} \mu_t\left(\mathbf{R} + \hat{\mathbf{u}}_k \ell\right) d\ell$$

Reconstruction in media with inhomogeneous $\mu_s(y, z)$ can be obtained if

$$\sum_{k=1}^{K} s_k = 0$$

How is the star transform obtained from the physical measurements $\phi_{ij}(\mathbf{R})$?

$$\Phi(\mathbf{R}) = \frac{1}{2}\sum_{j,k=1}^{K} c_{jk}\phi_{jk}(\mathbf{R})$$

Conditions on c_{jk}

(i) $c_{jk} = c_{kj}$

(ii) $c_{kk} = 0$

(iii) $\sum_{j,k=1}^{K} c_{jk} = 0$

(iv) $\sum_{j=1}^{K} c_{jk} = s_k$

$$\Phi(\mathbf{R}) = \sum_{k=1}^{K} s_k I_k(\mathbf{R})$$

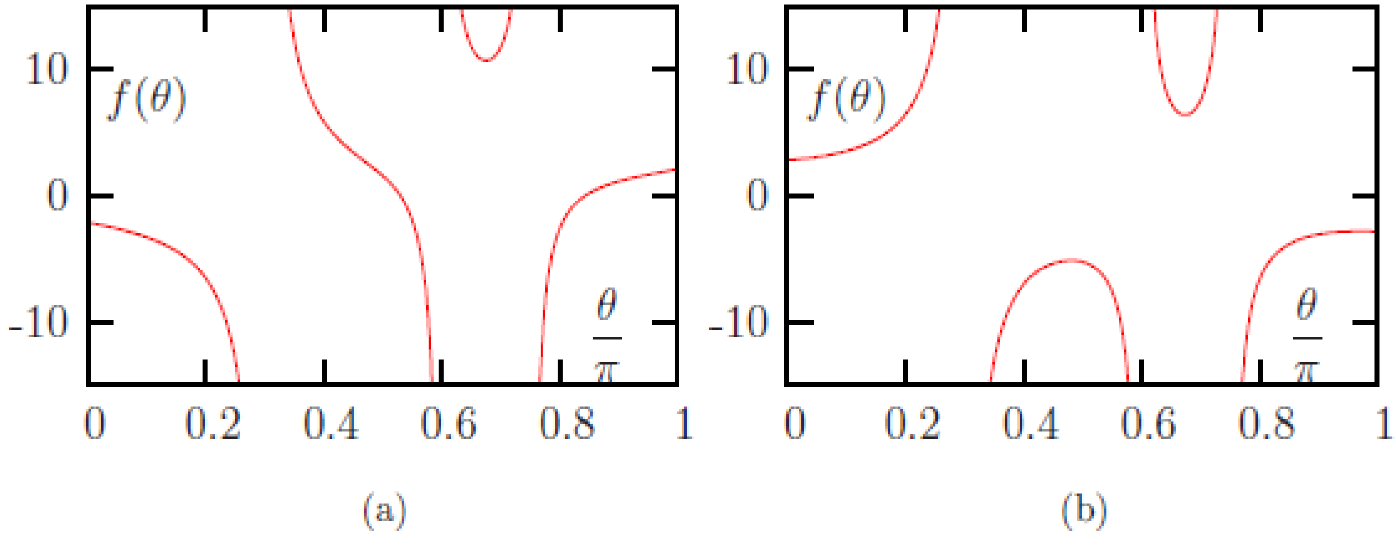
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & -2 & -1 \\ 1 & -2 & 0 & -1 \\ \hline 2 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 0 & -1 \\ \hline 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

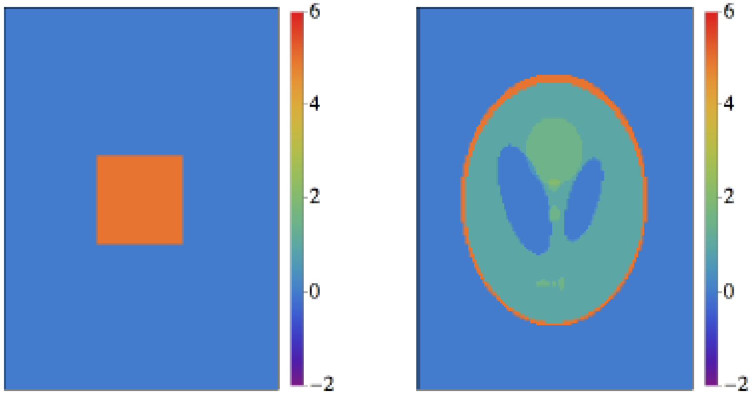
Analysis of Stability

- Analysis is done in Fourier space analytically at low and high spatial frequencies separately
- It can be shown that the transform is unstable if a certain trigonometric function has zeros.
- It can be shown that it always has zeros for even number of rays
- For odd K there is an additional condition on the unit vectors $\hat{\mathbf{u}}_k$ and coefficients s_k

$$f(\theta) = \sum_{k=1}^{K} \frac{s_k}{\cos(\theta - \theta_k)},$$



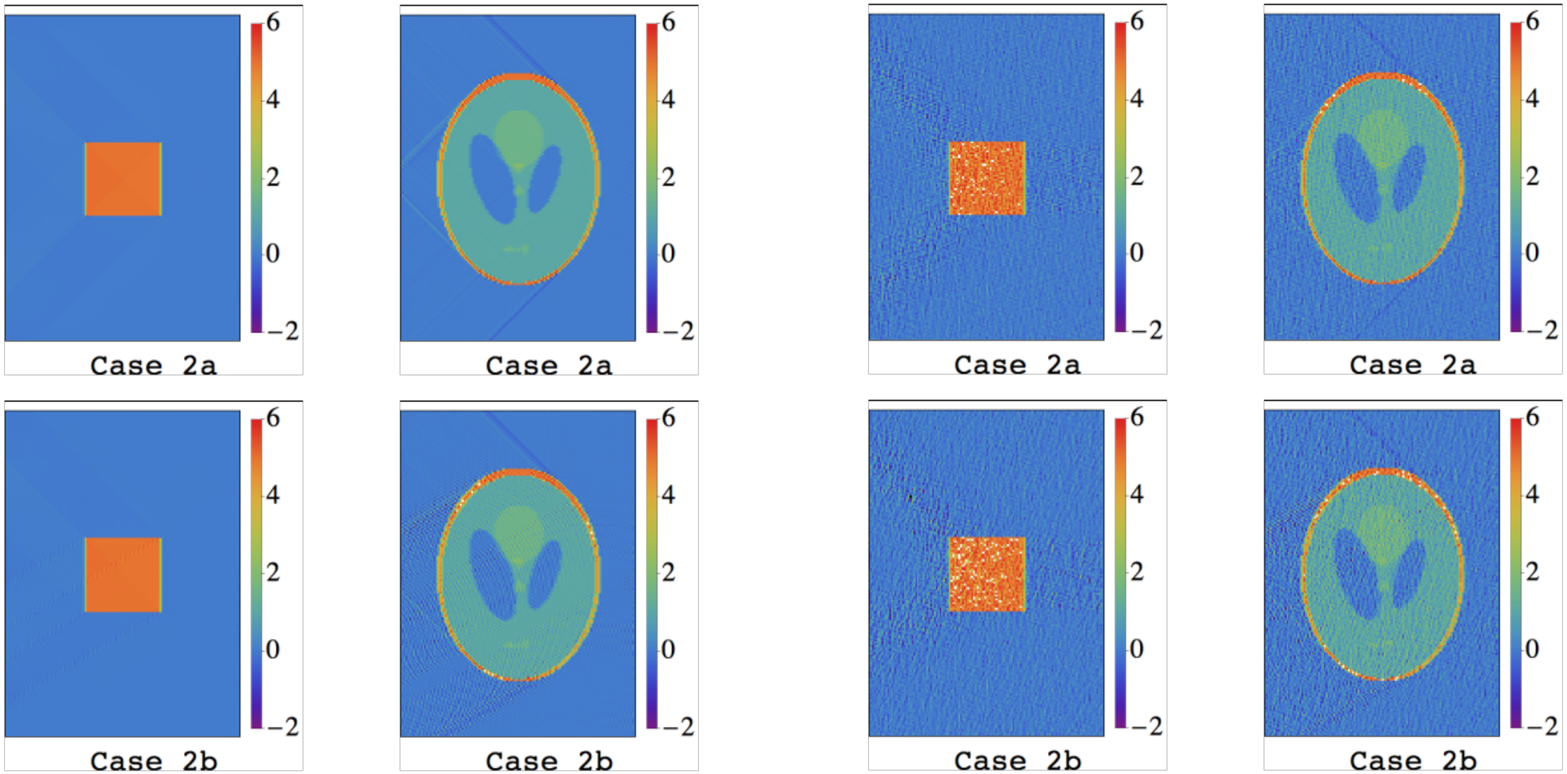
Numerical Examples



Phantoms used

From F. Zhao, J.C. Schotland and V.A.Markel,
Inverse Problems, in press, 2014
arXiv:1401.7655

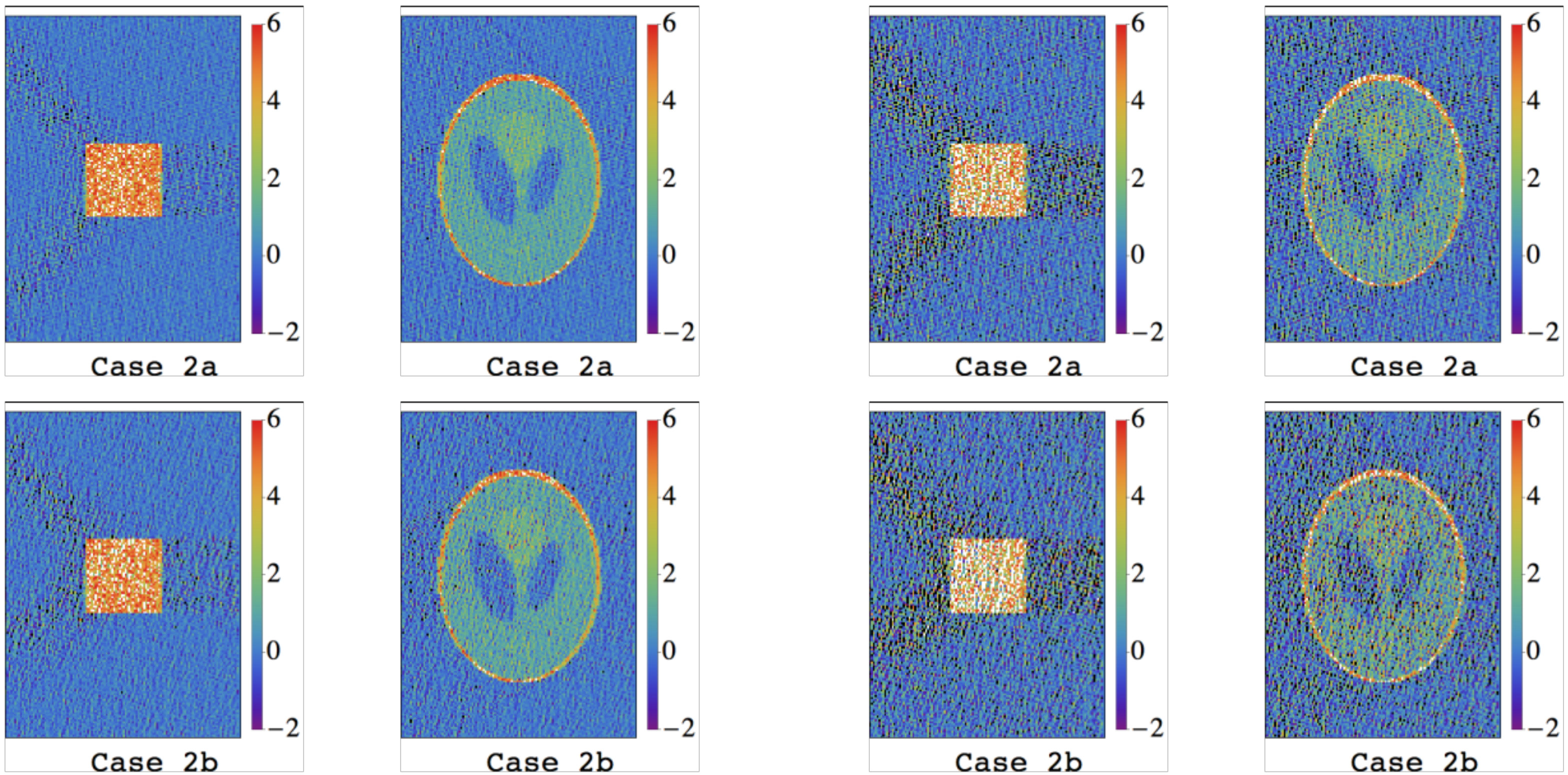
$K = 3;\ \ s_1 = s_2 = s_3 = 1;$ Reconstruction of total attenuation only



$\mathcal{N} = \infty$ (no noise)

$\mathcal{N} = 4 \cdot 10^4$

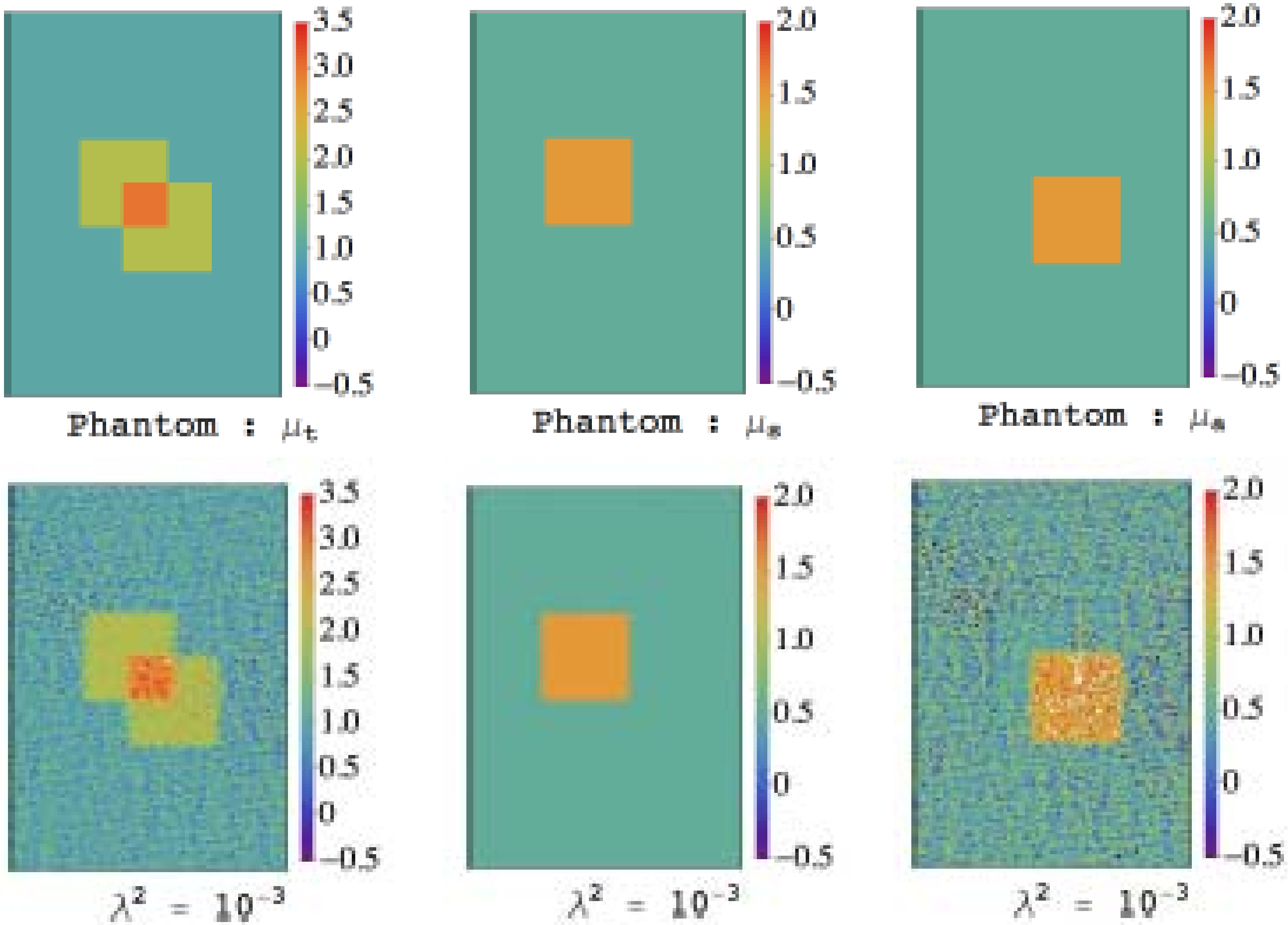
$\mathcal{N}$ is the number of incident photons per incident ray, which is used to determine the level of Poissonian noise in the data



$\mathcal{N} = 10^4$

$\mathcal{N} = 2.5 \cdot 10^3$

$K=3;\ \ s_1=s_2=1;\ s_3=-2;$ Simultaneous reconstruction of μ_a and μ_s



PHANTOM

Reconstruction with noise and Tikhonov regularization

Total attenuation

Scattering

Absorption

Local Methods

$$-\left(\hat{\mathbf{u}}_k \cdot \nabla\right) I_k\left(\mathbf{R}\right) = \mu\left(\mathbf{R}\right)$$

Unfortunately, we can not make measurements of ray integrals $I_k\left(\mathbf{R}\right)$ directly. However, we can formulate the star transfrom so that the coefficients $\mathbf{s}_k$ and $\mathbf{\Phi}$ are vectors. Then it is possible to invert the star transform by the local formula

$$\mu\left(\mathbf{R}\right) = \nabla \cdot \mathbf{\Phi}\left(\mathbf{R}\right)$$

[A. Katsevich and R. Krylov, Inverse Problems **29**, 075008, 2013]

To obtain local reconstruction methods,

we must allow the coefficients $\mathbf{c}_{jk}$ to be vectors!

Moreover, let

$$\sum_{j=1}^{K} \mathbf{c}_{jk} = \mathbf{s}_k = \sigma_k \hat{\mathbf{u}}_k \quad \text{and} \quad \sum_{k=1}^{K} \mathbf{s}_k = \sum_{k=1}^{K} \sigma_k \hat{\mathbf{u}}_k = 0$$

Then define

$$\boldsymbol{\Phi}(\mathbf{R}) \equiv \frac{1}{2} \sum_{j,k=1}^{K} \mathbf{c}_{jk} \phi_{jk}(\mathbf{R})$$

it then follows that

$$\mu(\mathbf{R}) = -\frac{1}{\sum_{k=1}^{K} \sigma_k} \nabla \cdot \boldsymbol{\Phi}(\mathbf{R})$$

Coefficient matrix (1)

$K = 3$

$$\left(\begin{array}{ccc|c} 0 & \sigma_1\hat{\mathbf{u}}_1+\sigma_2\hat{\mathbf{u}}_2 & \sigma_1\hat{\mathbf{u}}_1+\sigma_3\hat{\mathbf{u}}_3 & \sigma_1\hat{\mathbf{u}}_1 \\ \sigma_1\hat{\mathbf{u}}_1+\sigma_2\hat{\mathbf{u}}_2 & 0 & \sigma_2\hat{\mathbf{u}}_2+\sigma_3\hat{\mathbf{u}}_3 & \sigma_2\hat{\mathbf{u}}_2 \\ \sigma_1\hat{\mathbf{u}}_1+\sigma_3\hat{\mathbf{u}}_3 & \sigma_2\hat{\mathbf{u}}_2+\sigma_3\hat{\mathbf{u}}_3 & 0 & \sigma_3\hat{\mathbf{u}}_3 \\ \hline \sigma_1\hat{\mathbf{u}}_1 & \sigma_2\hat{\mathbf{u}}_2 & \sigma_3\hat{\mathbf{u}}_3 & 0 \end{array}\right)$$

Here σ_k are chosen so that $\sigma_1\hat{\mathbf{u}}_1+\sigma_2\hat{\mathbf{u}}_2+\sigma_3\hat{\mathbf{u}}_3=0$

Reconstruction formula:

$$\mu = -\frac{1}{\sigma_1+\sigma_2+\sigma_3}\nabla\cdot\left[\sigma_1\hat{\mathbf{u}}_1\left(\phi_{12}+\phi_{13}\right)+\sigma_2\hat{\mathbf{u}}_2\left(\phi_{21}+\phi_{23}\right)+\sigma_3\hat{\mathbf{u}}_3\left(\phi_{31}+\phi_{32}\right)\right]$$

Symmetric case $\hat{\mathbf{u}}_1+\hat{\mathbf{u}}_2+\hat{\mathbf{u}}_3=0$:

$$\mu = -\frac{1}{3}\nabla\cdot\left[\hat{\mathbf{u}}_1\left(\phi_{32}-\phi_{21}\right)+\hat{\mathbf{u}}_2\left(\phi_{31}-\phi_{12}\right)\right]$$

Coefficient matrix (2)

$K = 4$ (original method of Katsevich and Krylov)

$\sigma_1 = 0, \sigma_2 = 1, \sigma_3 = -a, \sigma_4 = -b$

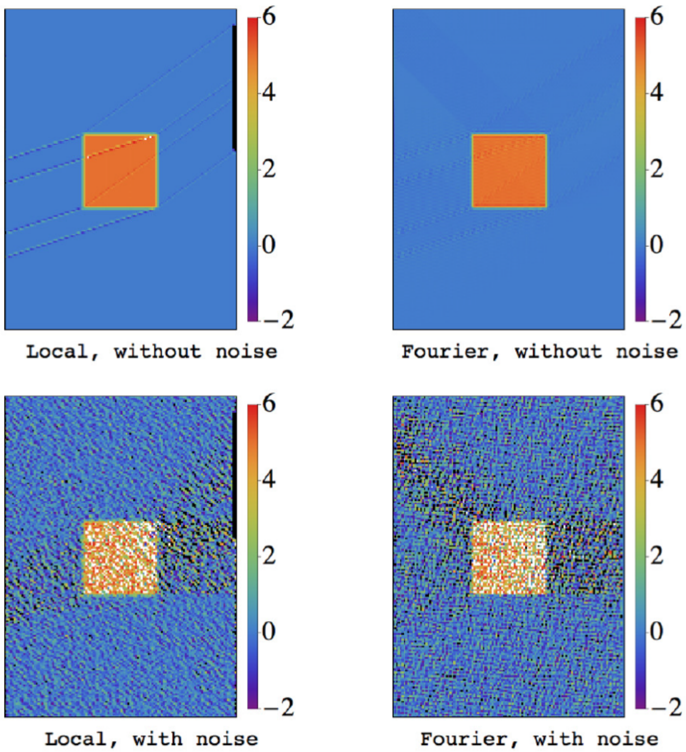
$$\hat{\mathbf{u}}_2 - a\hat{\mathbf{u}}_3 - b\hat{\mathbf{u}}_4 = 0$$

$$\left[\begin{array}{cccc|c} 0 & \hat{\mathbf{u}}_2 & -a\hat{\mathbf{u}}_3 & -b\hat{\mathbf{u}}_4 & 0 \\ \hat{\mathbf{u}}_2 & 0 & 0 & 0 & \hat{\mathbf{u}}_2 \\ -a\hat{\mathbf{u}}_3 & 0 & 0 & 0 & -a\hat{\mathbf{u}}_3 \\ -b\hat{\mathbf{u}}_4 & 0 & 0 & 0 & -b\hat{\mathbf{u}}_4 \\ \hline 0 & \hat{\mathbf{u}}_2 & -a\hat{\mathbf{u}}_3 & -b\hat{\mathbf{u}}_4 & 0 \end{array}\right]$$

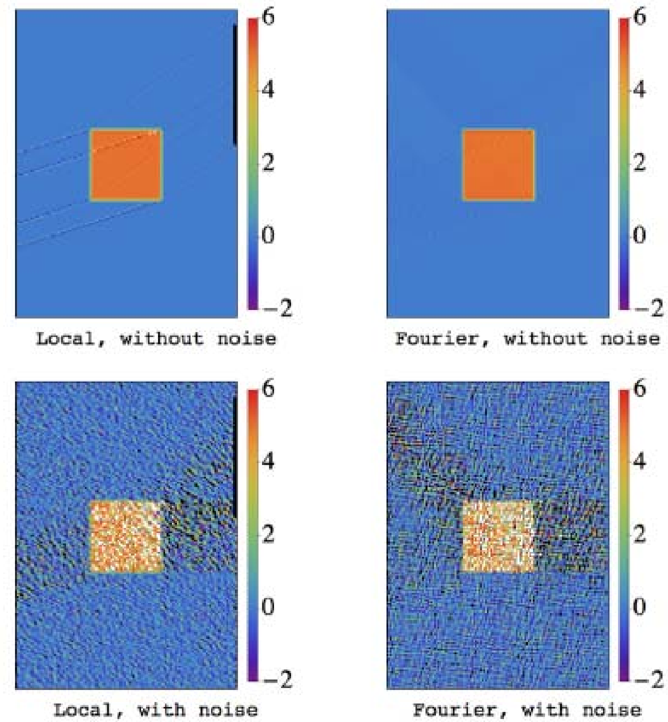
Reconstruction formula:

$$\mu = \frac{1}{a+b-1} \nabla \cdot \left[\hat{\mathbf{u}}_2 \phi_{12} - a\hat{\mathbf{u}}_3 \phi_{13} - b\hat{\mathbf{u}}_4 \phi_{14}\right]$$

Comparison of local and global (Fourier –space) methods :
Similar quality BUT local methods can work with limited data
(this last property is extremely important)

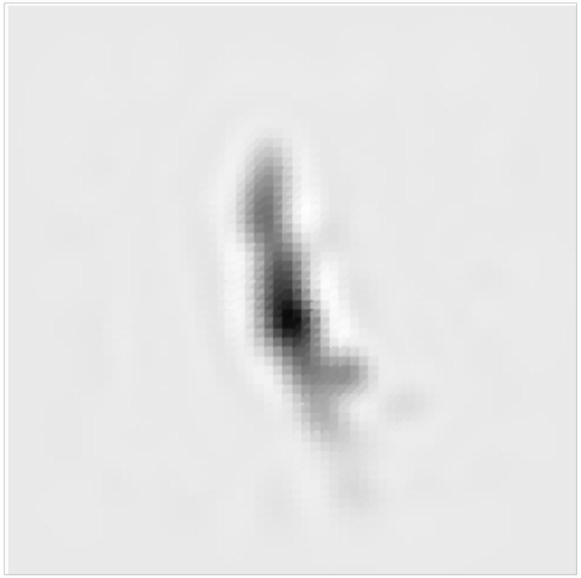
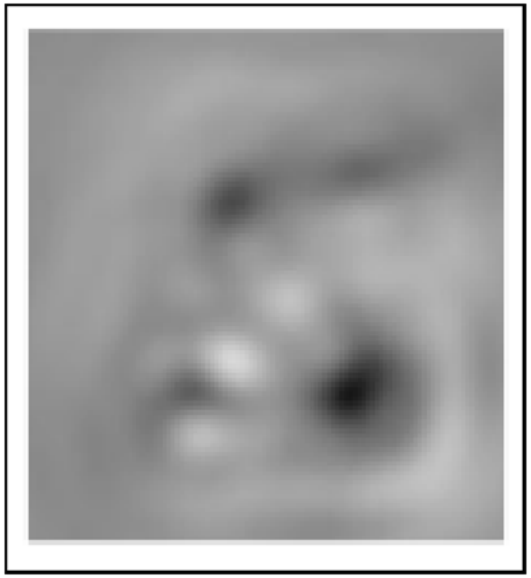
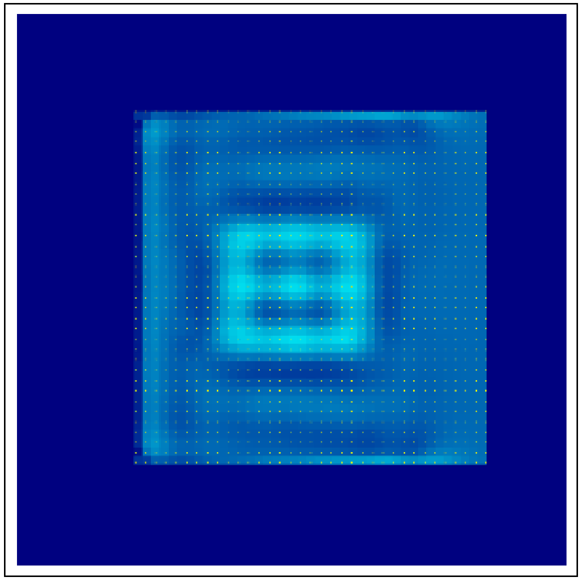


K=3 star; coefficient matrix (1)

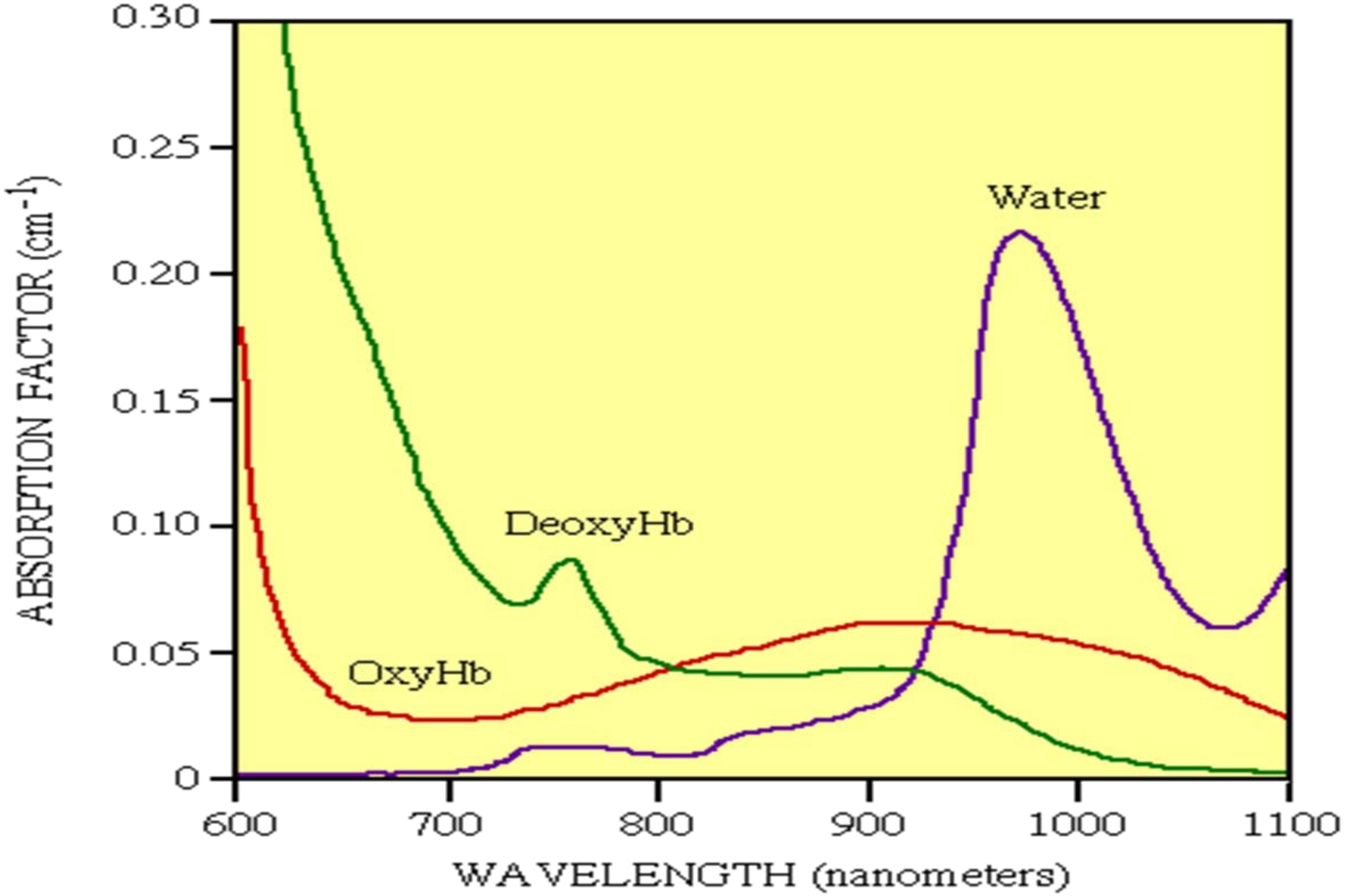


K=4 star with $\mathbf{s}_1$=0; coefficient matrix (2)
(original method of Katsevich and Krylov)

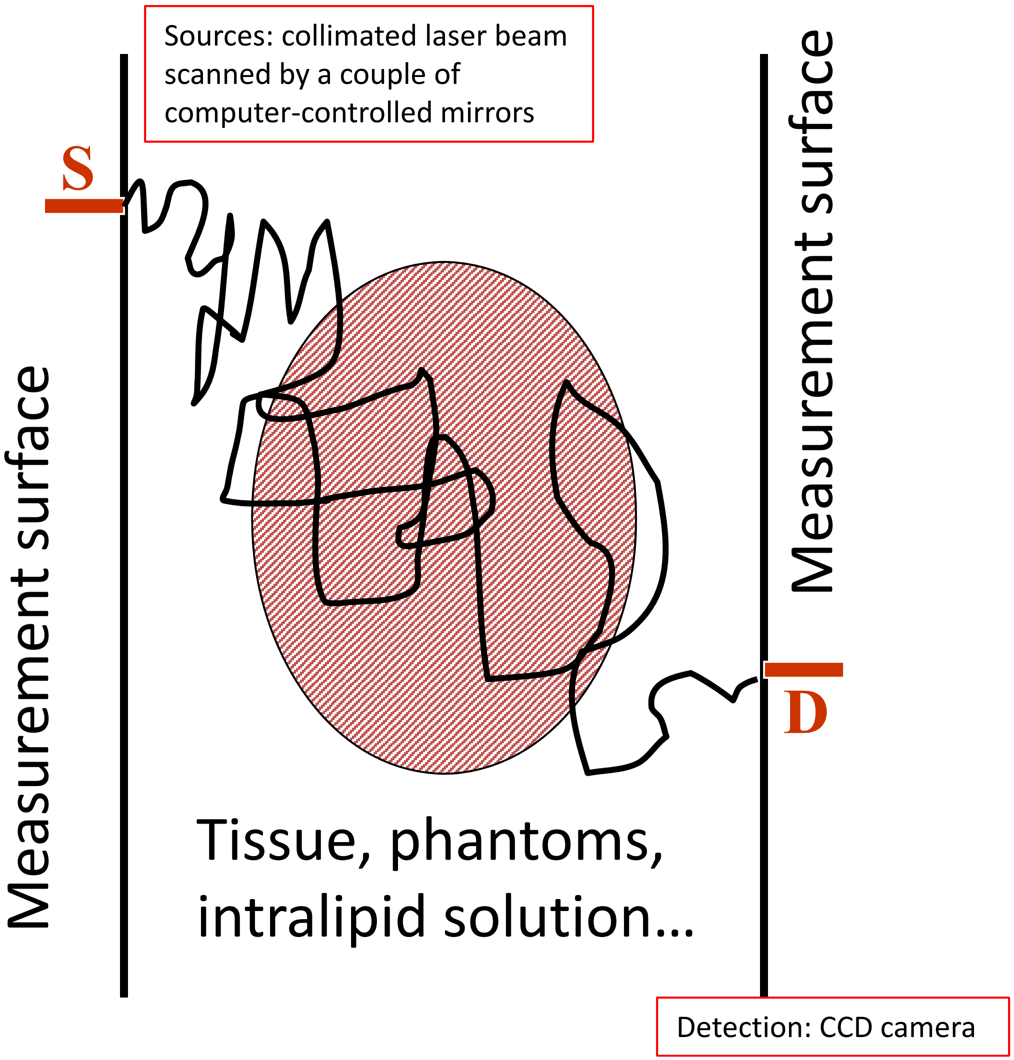
2. STRONG SCATTERING REGIME (ODT)

Absorption Characteristics of Water and Blood in the Near Infra-Red



What is Optical Diffusion Tomography (ODT)?



Problem: given multiple measurements with different near-IR sources (S) and detectors (D), obtain 3D images of tissue (absorption coefficient, scattering coefficient, …)

S and D are not necessarily in the same slice!

Physical Model and Contrast Mechanism (Diffusion Approximation)

Current of EM energy

Density of EM energy

$$\mathbf{J} = -D\nabla u \text{ (accurate if } \mu_s \gg \mu_a \text{) ;}$$

Absorption coefficient

$$\alpha = c\mu_a \ ; \quad D = \frac{c}{3[\mu_a + (1-g)\mu_s]}$$

Diffusion coefficient

Scattering asymmetry parameter

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u + \alpha u = S$$

Time-dependent diffusion equation with absorption

Source

More Details on Diffusion Equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t} - \nabla \cdot D\nabla u(\mathbf{r},t) + \alpha u(\mathbf{r},t) = S(\mathbf{r},t)$$

Mixed boundary conditions at diffuse-nondiffuse inter- faces:

$$\left(u + \ell \hat{\mathbf{n}} \cdot \nabla u\right)\Big|_{\mathbf{r}\in\text{boundary}} = 0$$

$$\ell = 0 \quad : \text{purely absorbing boundaries}$$

$$\ell = \infty \quad : \text{purely reflecting boundaries}$$

Measurable signal:

$$I\left(\mathbf{r}, \hat{\mathbf{s}}\right) = \frac{c}{4\pi}\left(u + \ell^* \hat{\mathbf{s}} \cdot \nabla u\right) \approx \text{const} \times u\left(\mathbf{r}\right)\Big|_{\mathbf{r}\in\text{boudary},\hat{\mathbf{s}}=\hat{\mathbf{n}}}$$

$$\ell^* = \frac{3D}{c}$$

Typical Values of Constants in Human Tissues

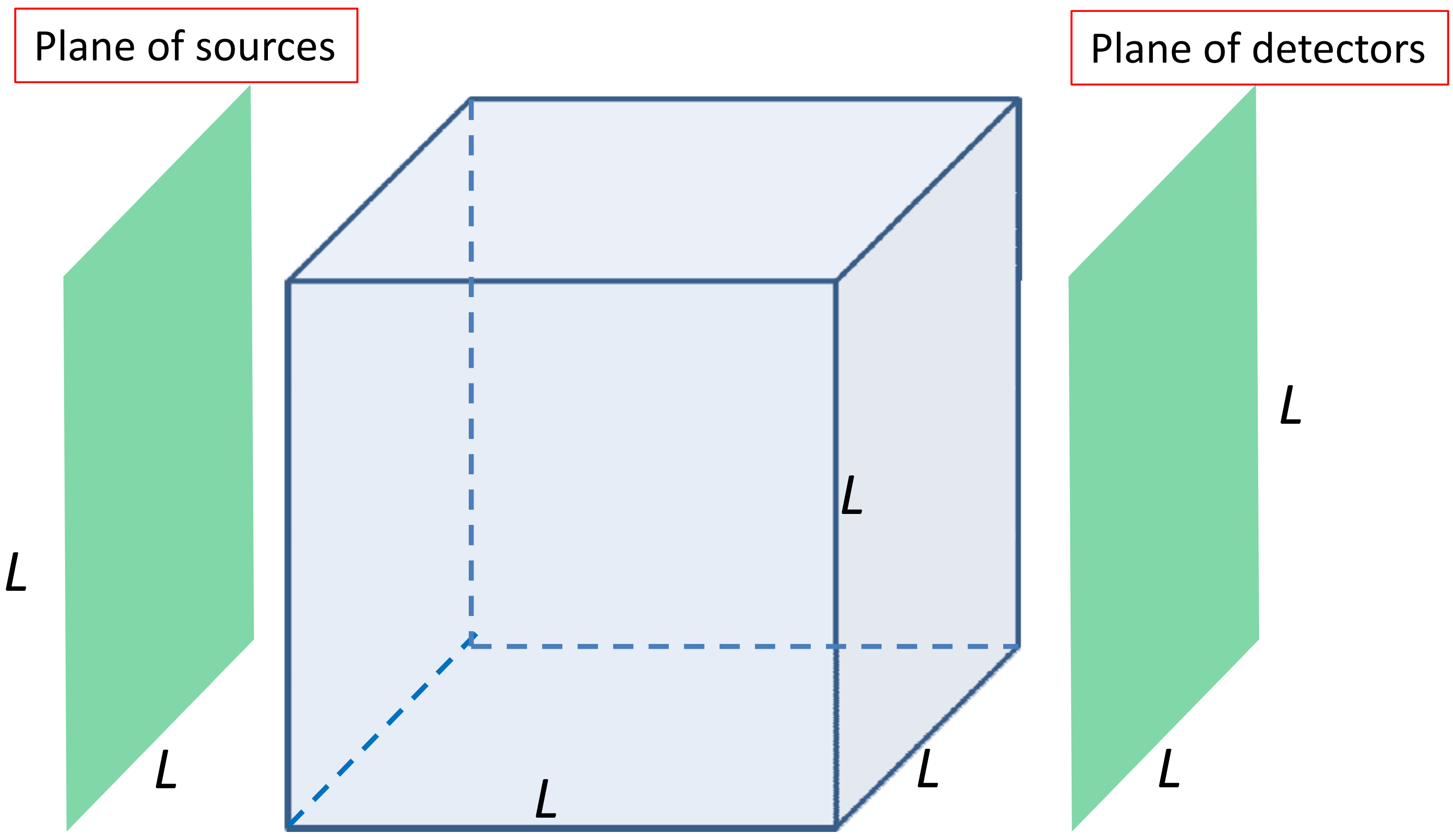
$$\lambda = 800nm$$

$$\ell^* = 1mm$$

$$D = \frac{(c/n)}{3}\ell^* = \frac{1\ \text{cm}^2}{\text{nsec}}$$

$$\alpha = 1\ \text{nsec}^{-1} = 1\ \text{GHz}$$

$$k_{\text{diff}} = \sqrt{\frac{\alpha}{D}} = 2\pi / \lambda_{\text{diff}} = 1\text{cm}^{-1}$$


Plane of sources
Plane of detectors
L
L
L
L
L
L
L
L

Number of sources : $N_s = L^2$

Number of detectors : $N_d = L^2$

Number of measurements : $N_m = N_s N_d = L^4$

Number of voxels : $N_v = L^3$

If $L = 100$, the problem size is $10^8 \times 10^6$

The inverse problem of ODT is typically
- strongly overdetermined
- severely ill-posed

Should we use strongly over determined datasets?

Develop numerical methods to utilize very large data sets

The typical approach to inverse problem is to write the DE in the integral form:

Green's function for the simplest case of CW illumination, constant diffusion coefficient:

$$\left(\nabla^2 - \frac{\alpha_0 + \delta\alpha(\mathbf{r})}{D_0}\right) G(\mathbf{r}, \mathbf{r}') = -\frac{1}{D_0}\delta(\mathbf{r} - \mathbf{r}')$$

$$G(\mathbf{r}_d, \mathbf{r}_s) = G_0(\mathbf{r}_d, \mathbf{r}_s) - \int_V G_0(\mathbf{r}_d, \mathbf{r})\delta\alpha(\mathbf{r})G(\mathbf{r}, \mathbf{r}_s)\,\mathrm{d}^3 r$$

$$I(\mathbf{r}_d, \mathbf{r}_s) = A_d G(\mathbf{r}_d, \mathbf{r}_s) A_s \quad \leftarrow \text{ Measurable signal}$$

Coupling constants (sometimes can be eliminated)

The inverse problem is nonlinear because $G(\mathbf{r}, \mathbf{r}')$ itself depends on $\delta\alpha$

Linearizing approximations:

- First Born
- First Rytov
- Mean field

Linearized Integral Equation

First Born approximation: $G(\mathbf{r},\mathbf{r}_s) \rightarrow G_0(\mathbf{r},\mathbf{r}_s)$
(inside the integral only!)

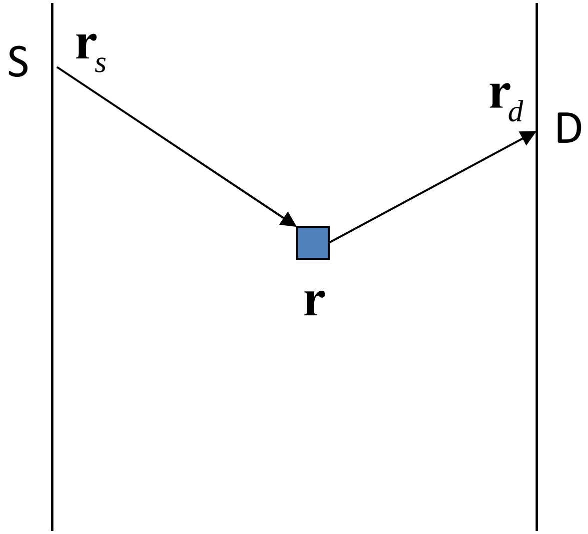
Then define the data function as

$$\phi(\mathbf{r}_s,\mathbf{r}_d) = \frac{I_0(\mathbf{r}_s,\mathbf{r}_d) - I(\mathbf{r}_s,\mathbf{r}_d)}{I_0(\mathbf{r}_s,\mathbf{r}_d)} G_0(\mathbf{r}_s,\mathbf{r}_d)$$

Then we obtain linear int. equation:

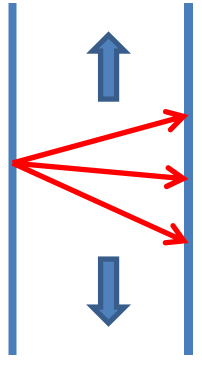
$$\int \Gamma(\mathbf{r}_d,\mathbf{r}_s;\mathbf{r})\delta\alpha(\mathbf{r})d^3r = \phi(\mathbf{r}_d,\mathbf{r}_s)$$

$$\Gamma(\mathbf{r}_d,\mathbf{r}_s;\mathbf{r}) = G_0(\mathbf{r}_d;\mathbf{r})G_0(\mathbf{r};\mathbf{r}_s)$$



“Analytical” (fast) SVD Approach

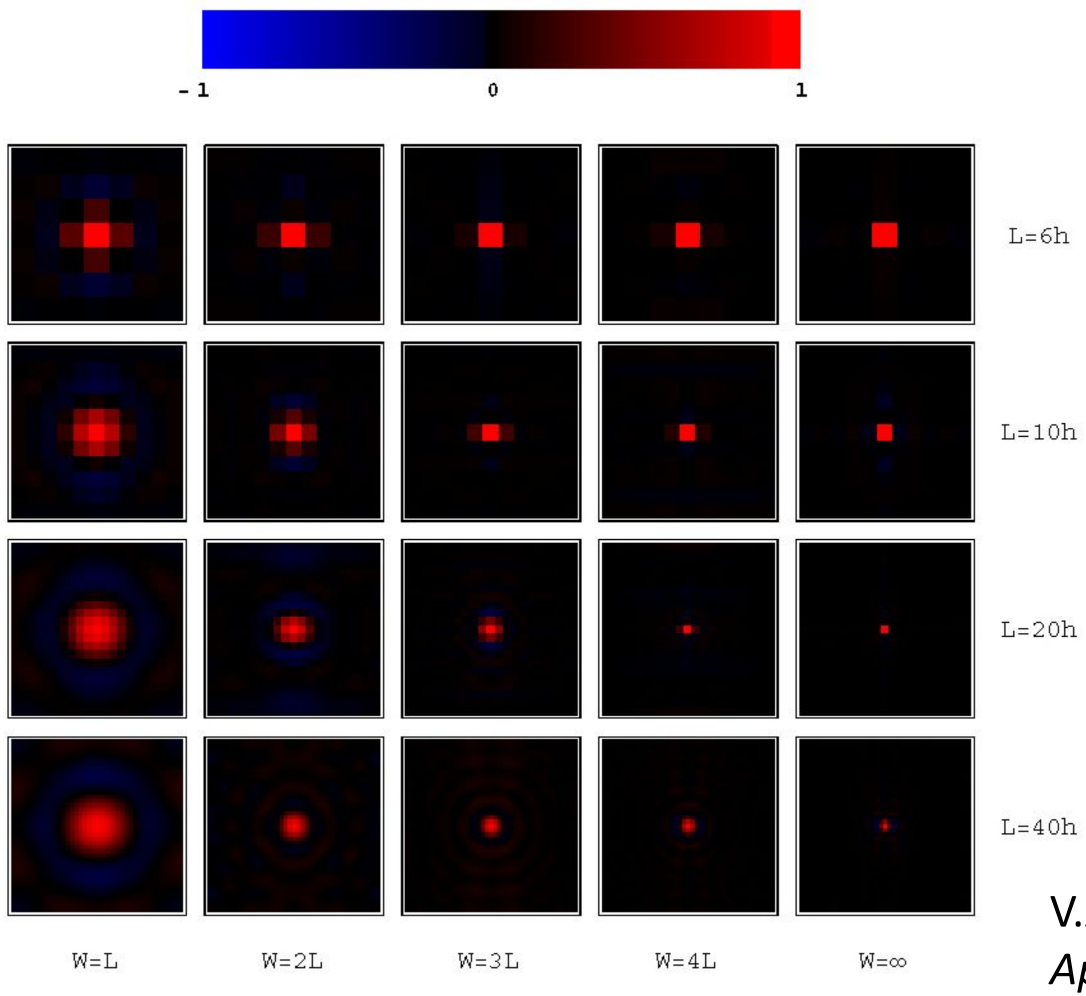
• Takes advantage of the translational invariance of the unperturbed medium
• Requires the sources to be on a lattice and a similarly situated set of detectors (relative to a given source) to be used for each source. This included detectors on a lattice. (Here “sources” and “detectors” can be interchanged.)



(i) FT transform the equation in the direction parallel to the slab.
(ii) In Fourier space, solve a 1D linear problem for each value of the Fourier variable **q** .
(iii) Then FT solution back to real space.

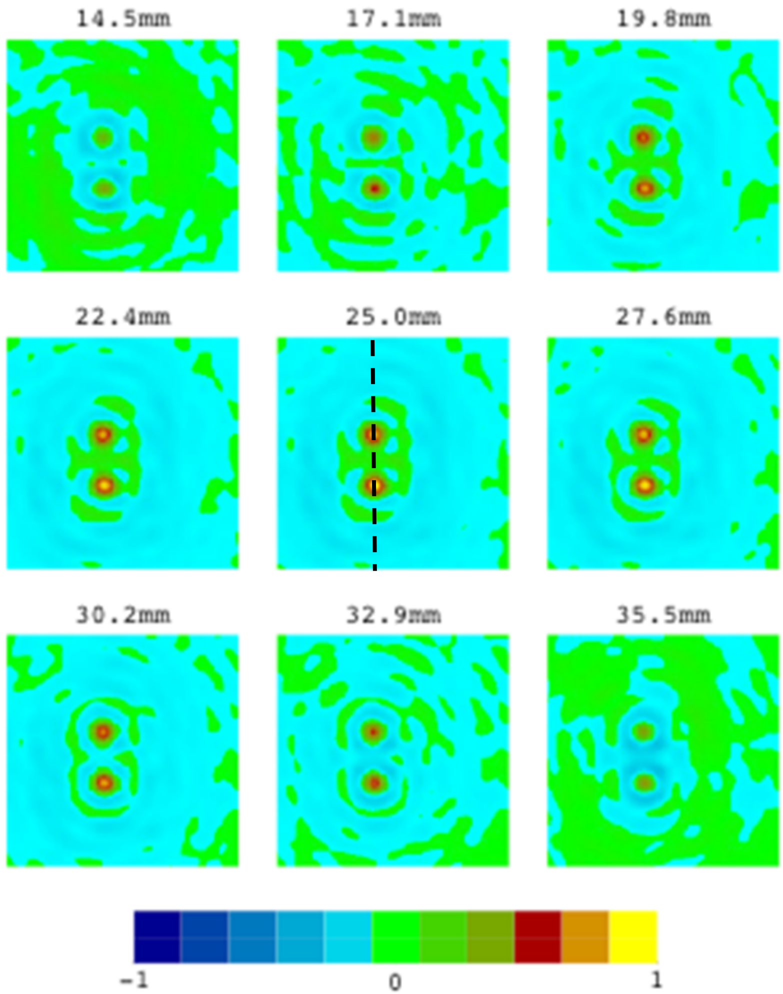
Why do we want a large number of sources and detectors?

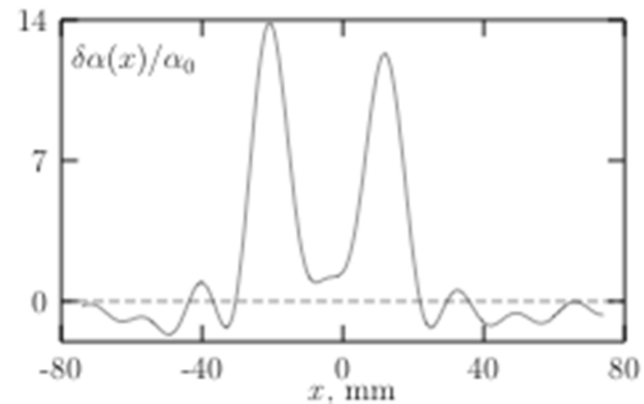
The fundamental resolution limit (in the plane) is equal to the spacing of sources/detectors, h



V.A.Markel and J.C.Schotland
Appl. Phys. Lett. **81**, 1180, 2002.

First Experimental Demonstration of the Fast Reconstruction Method for Two Black Balls in 5cm-Thick Slab Filled with Intralipid



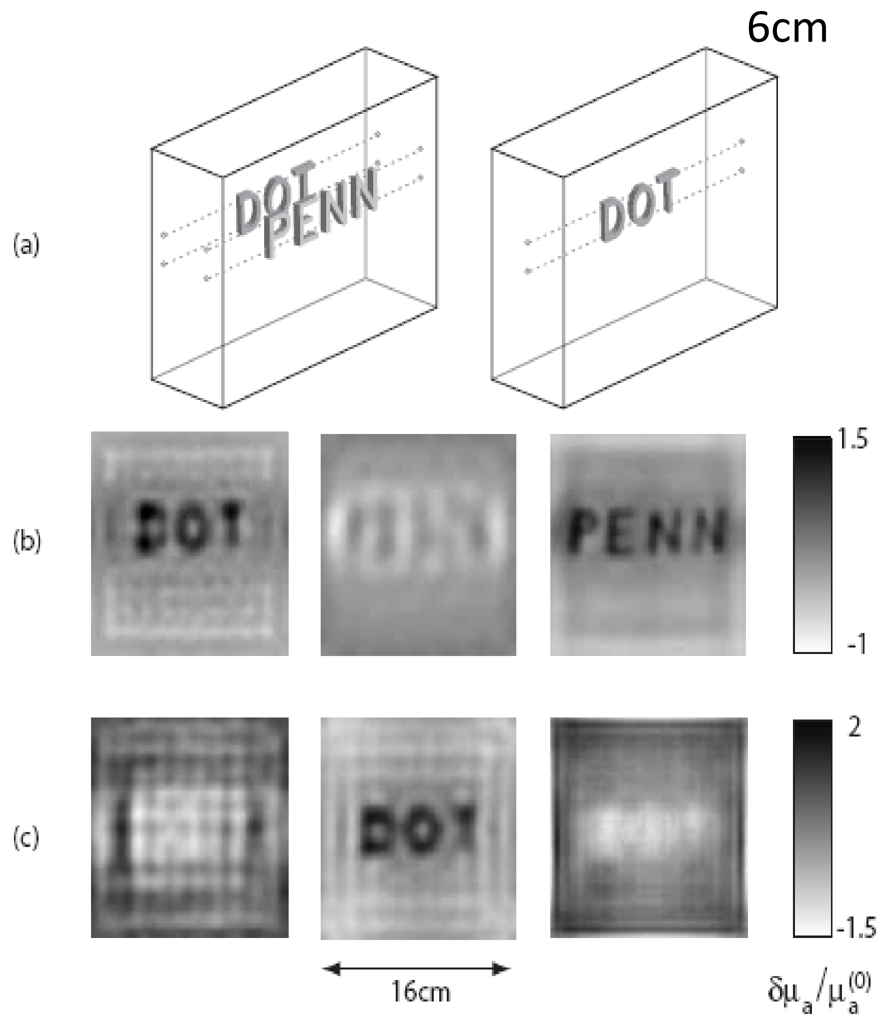


FWHMxy=11 mm

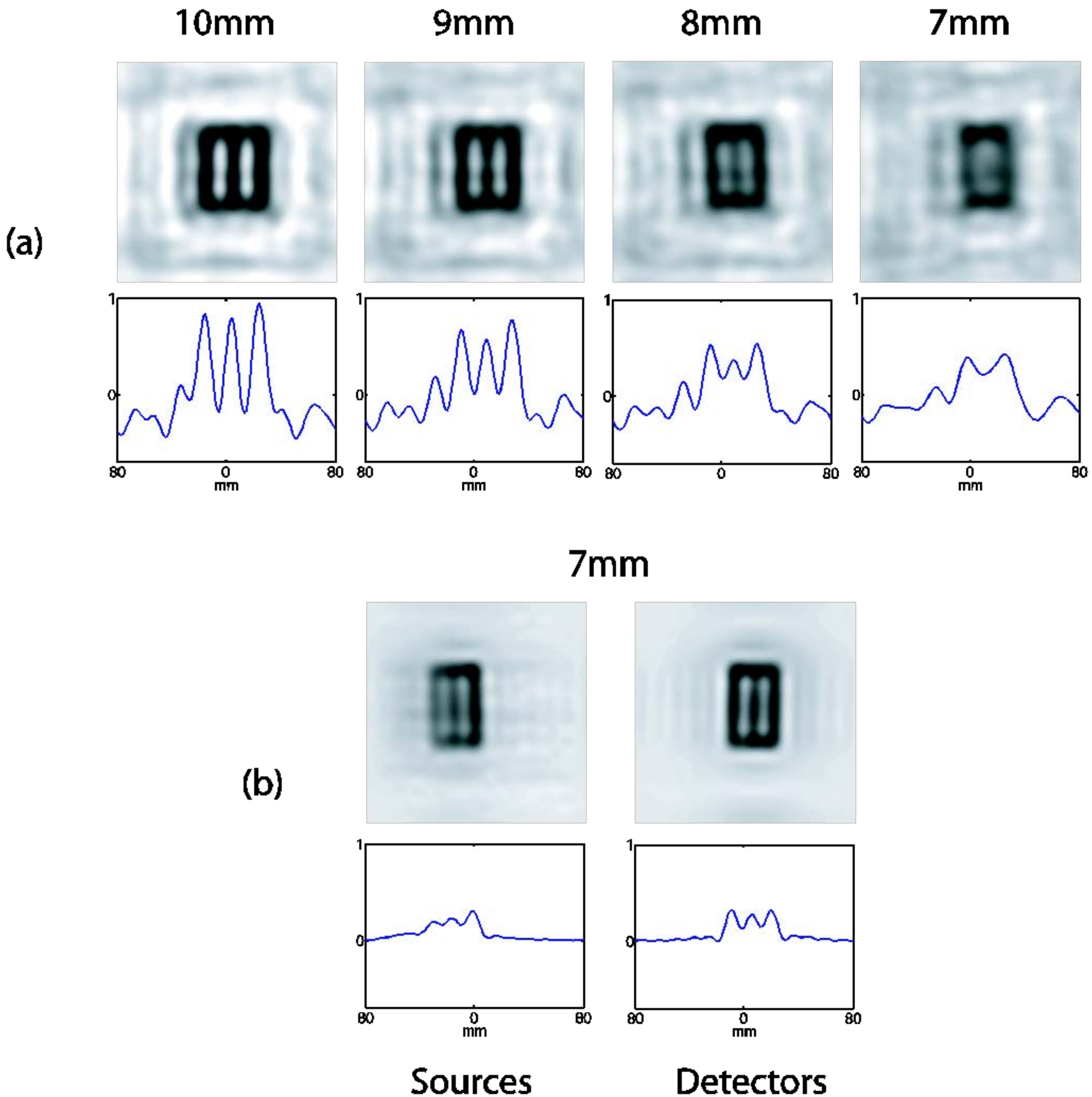
FWHMz=1.5 cm

Separation=3.3 cm

Z.M. Wang, G.Y.Panasyuk, V.A.Markel and J.C.Schotland, *Opt. Lett.* **30**, 3338, 2005.

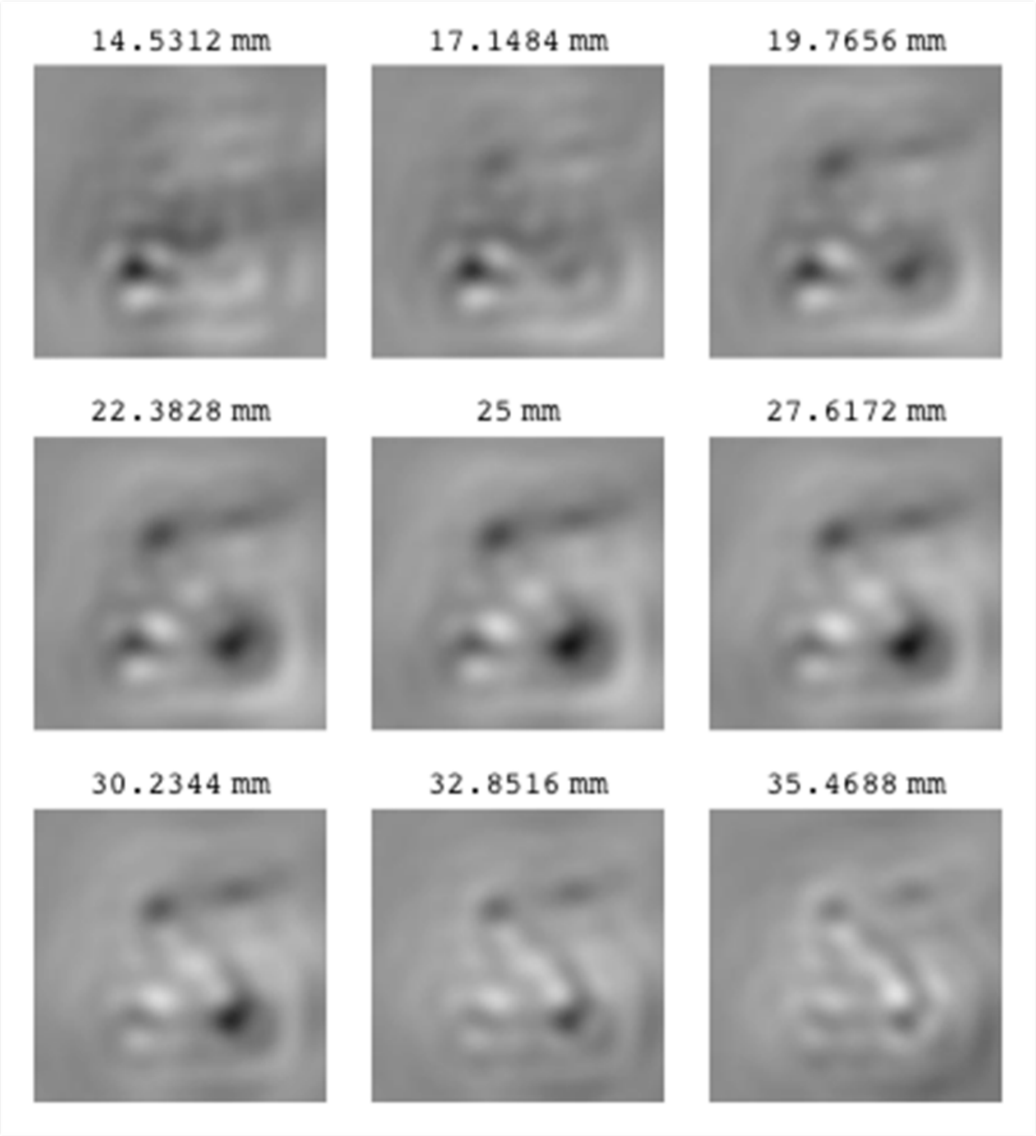


S.D.Konecky, G.Y.Panasyuk, K.Lee, V.Markel, A.G.Yodh and J.C.Schotland
Optics Express **16**, 5048, 2008.



10mm
9mm
8mm
7mm
(a)
1
0
80
0
mm
80
7mm
(b)
Sources
Detectors

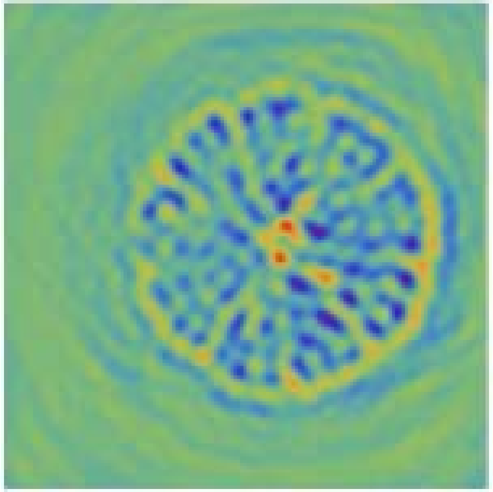
Reconstruction of a chicken wing

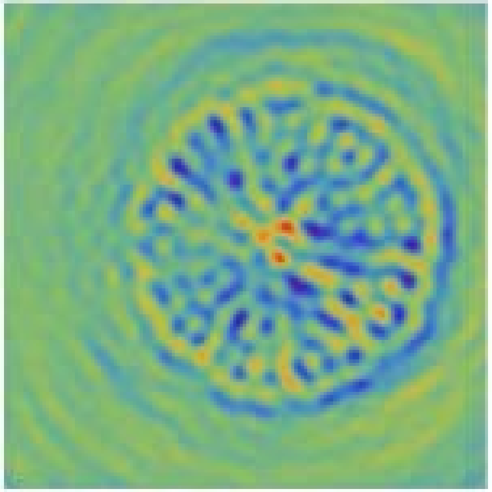


- 10^8 source-detector pairs
- 10^3 sources and 10^5 detectors
- 2.6 mm slice separation
- 15 cm x 15 cm FOV

3. INTERMEDIATE SCATTERING REGIME

Inverse solution of radiative transport equation by the method of rotated reference frames





Meet the RTE

$$(\hat{\mathbf{s}}\cdot\nabla+\mu_t)I(\mathbf{r},\hat{\mathbf{s}})=\mu_s\int A(\hat{\mathbf{s}},\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}')\mathrm{d}^2\hat{\mathbf{s}}'+\varepsilon(\mathbf{r},\hat{\mathbf{s}})$$

Transport operator: $(\hat{\mathbf{s}}\cdot\nabla+\mu_t)$ — Scattering operator: $\mu_s\int A(\hat{\mathbf{s}},\hat{\mathbf{s}}')$ — Source: $\varepsilon(\mathbf{r},\hat{\mathbf{s}})$

- The specific intensity *I(r,s)* is a function of position and direction
- *A(s,s')* is the scattering "phase function" (indicatrix of scattering)
- Transport operator and scattering operator do not commute
- Not solvable by separation of variables even in simple geometries
- Is a 5-dimensional equation
- Analytical solutions are not known even in simple geometries. Only in the case of strong (isotropic) scattering A(s,s')=const, an analytical solution is available in infinite space (no boundaries).
- In the case of strong scattering, RTE allows for a simplification by means of diffusion approximation.
- In the opposite case, scattering-order expansion can be used.

Can we use the fast "analytical" image reconstruction algorithm with RTE the same way we did with DE?

We would need the plane-wave decomposition of the Greens function, of the form

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}') = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g(\mathbf{q}; z,\hat{\mathbf{s}}; z',\hat{\mathbf{s}}') e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}$$

and the linearized integral kernel will be (in Fourier space)

$$\Gamma(\mathbf{q},\mathbf{p};z) = \int g(\mathbf{q}; z_s,\hat{\mathbf{z}}; z,\hat{\mathbf{s}}) g(\mathbf{p}; z,\hat{\mathbf{s}}; z_d,\hat{\mathbf{z}}) \mathrm{d}^2 s$$

The Conventional Method of Spherical Harmonics

This relmsults in the following system of equations with respect to the vector of the expansion coefficients $\left| I(\mathbf{k}) \right\rangle$ ($\mathbf{k}$ - the Fourier variable) :

$$iA_x k_x \left| I(\mathbf{k}) \right\rangle + iA_y k_y \left| I(\mathbf{k}) \right\rangle + iA_z k_z \left| I(\mathbf{k}) \right\rangle + D \left| I(\mathbf{k}) \right\rangle = \left| \varepsilon(\mathbf{k}) \right\rangle$$

$A_x, \ A_y, \ A_z, D$ are different matrices.

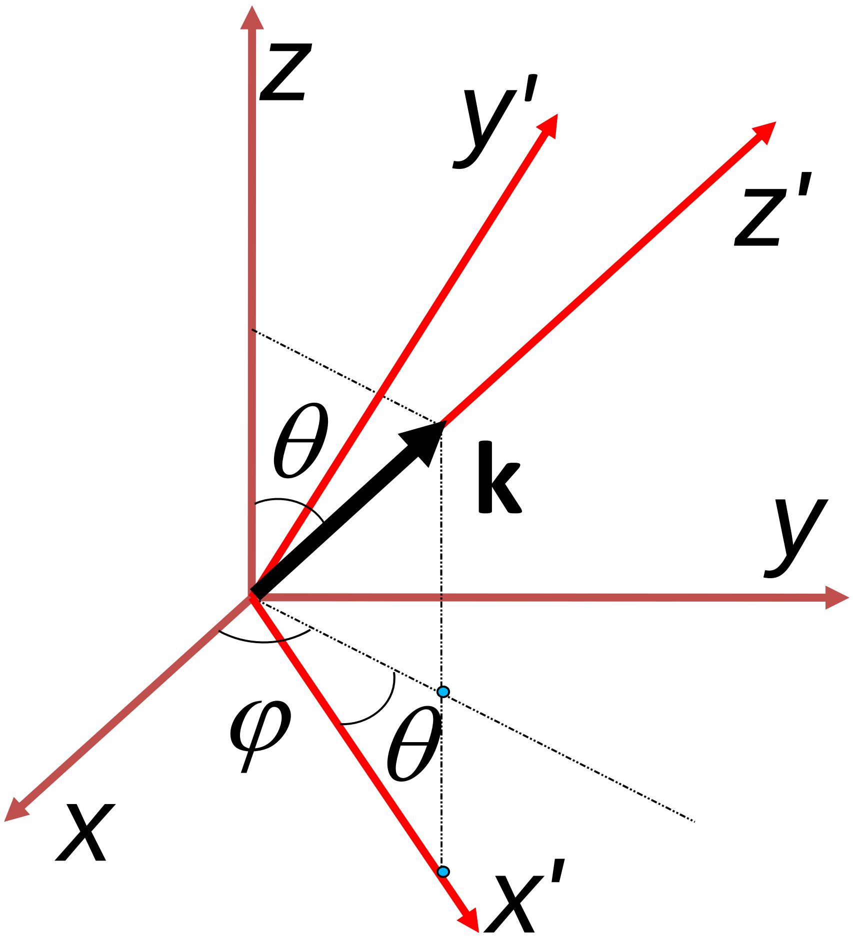
$$\left\langle lm \right| A_x \left| l'm' \right\rangle = \int \sin\theta \cos\varphi Y^*_{lm}(\theta,\varphi) Y_{l'm'}(\theta,\varphi) \sin\theta d\theta d\varphi \ , \quad \text{etc. .....}$$

"This rather awe-inspiring set of equations … has perhaps only academic interest".

K.M.Case, P.F.Zweifel, Linear Transport Theory

Method of Rotated Reference Frames:

Use spherical functions in special (rotated) reference frames whose Z-axis is collinear with the Fourier vector **k** in the spatial FT



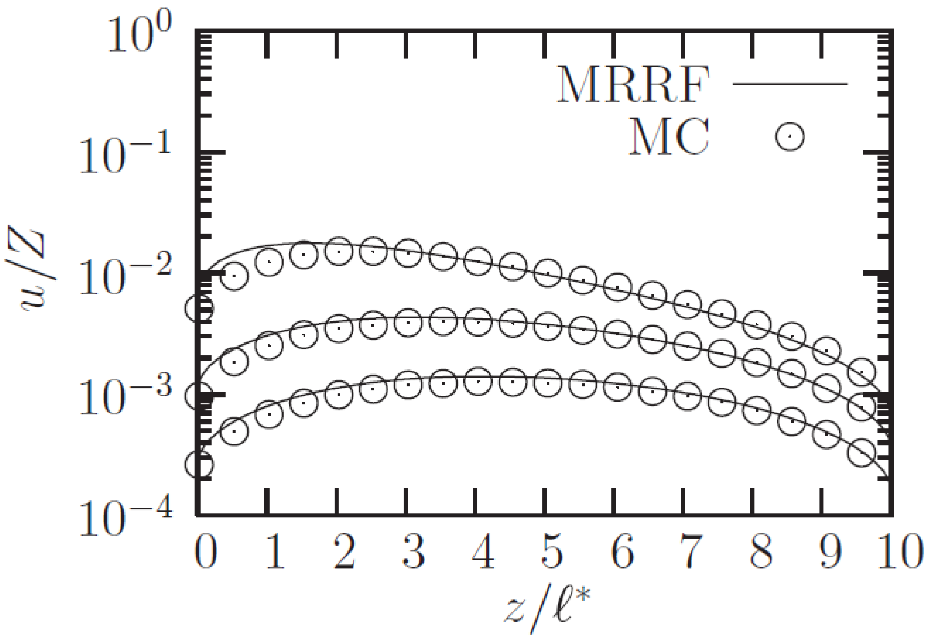
$$Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}}) = \sum_{m'=-l}^{l} D^{l}_{m'm}(\varphi_{\mathbf{k}},\theta_{\mathbf{k}},0)Y_{lm'}(\hat{\mathbf{s}})$$

Wigner D-functions

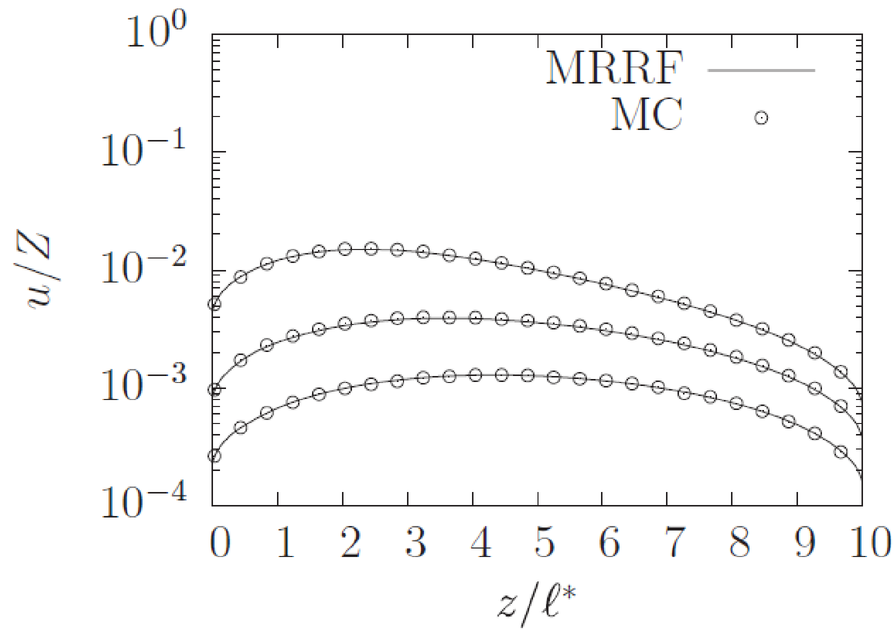
Euler angles

Spherical functions in the laboratory frame

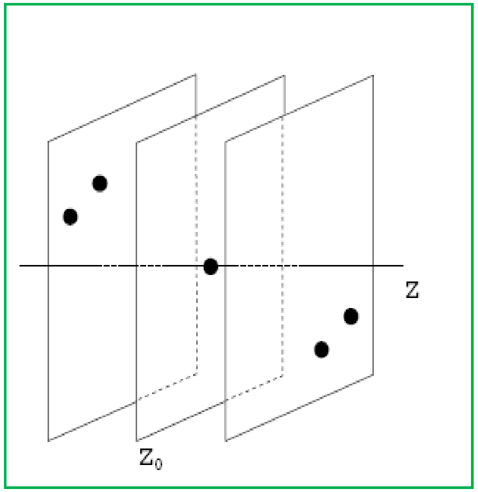
Comparison with Monte-Carlo



M. Machida, G. Y Panasyuk, J.C. Schotland and V. A. Markel, J. Phys. A **43,** 065402, 2010.

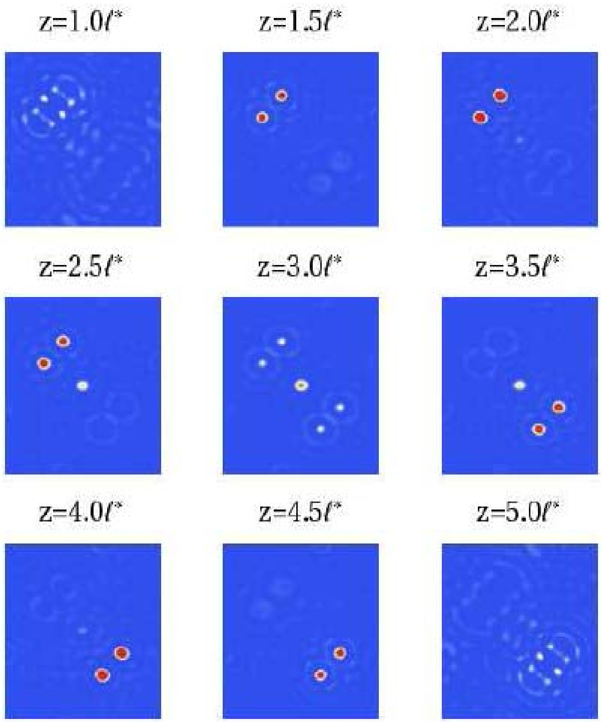


Corrigendum:
J. Phys. A **45**, 459501, 2012

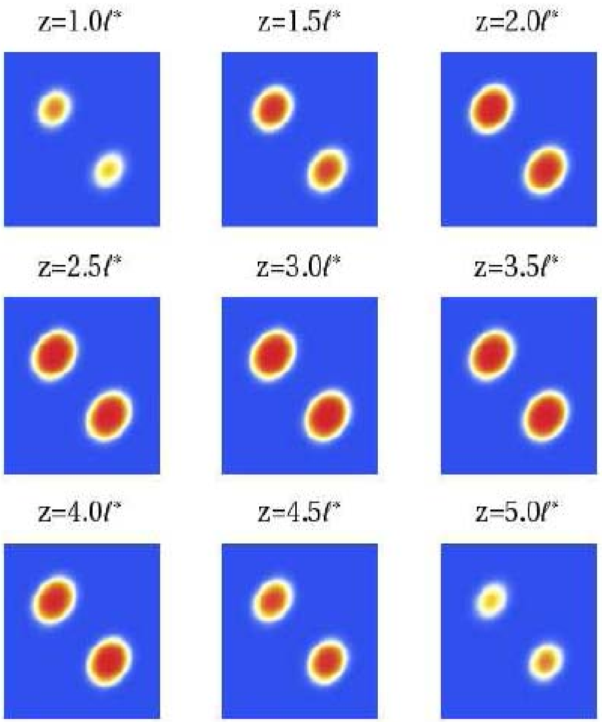


Simulations: Reconstruction of small absorbers

A set of 5 point absorbers in an $L = 6\ell^*$ slab.
The field of view is $16\ell^* \times 16\ell^*$.

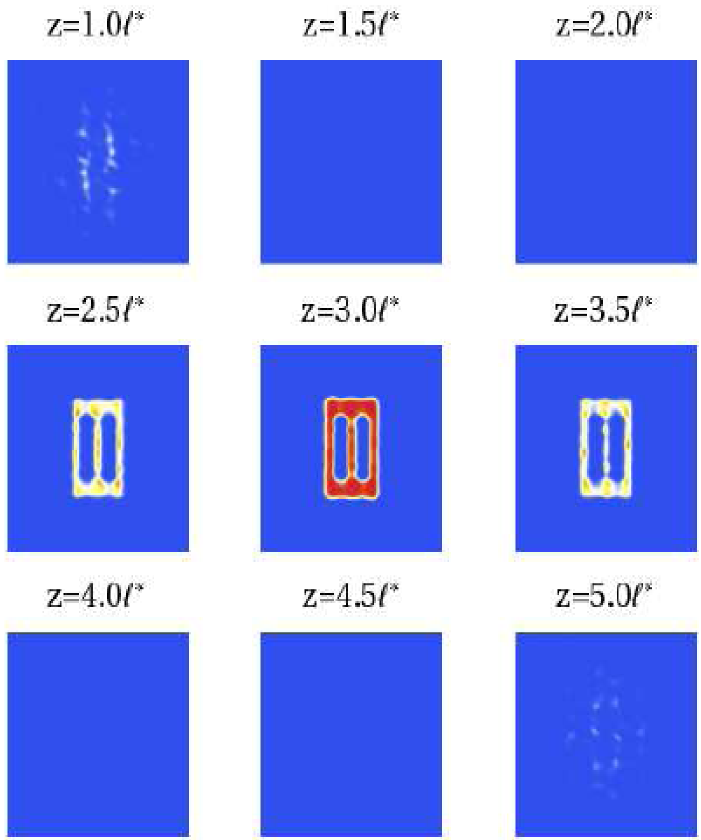


RTE

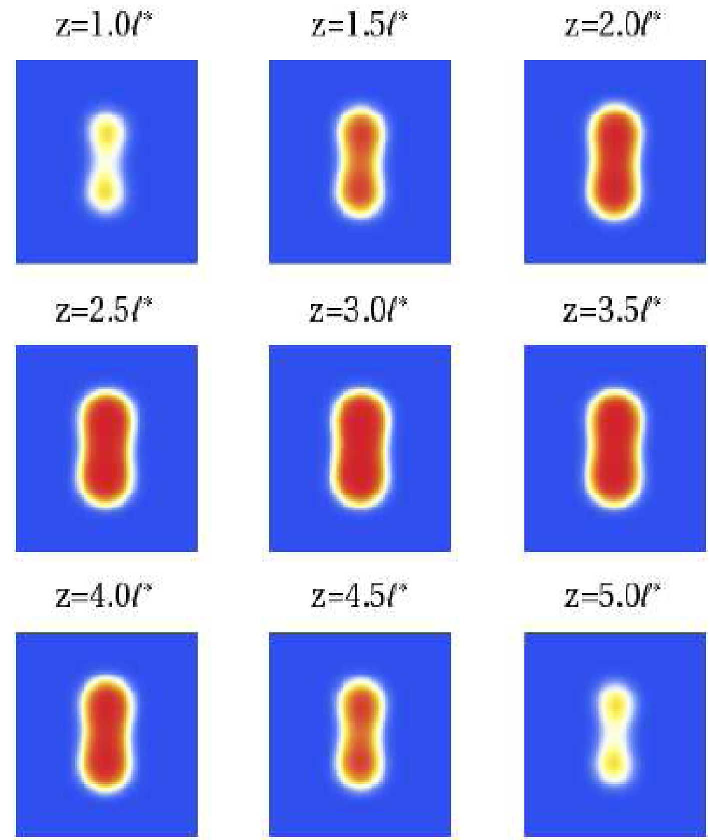


Diffusion approximation

A bar target in the center of the same slab

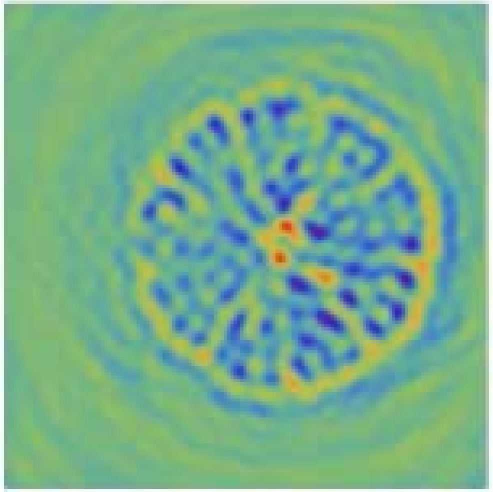


RTE

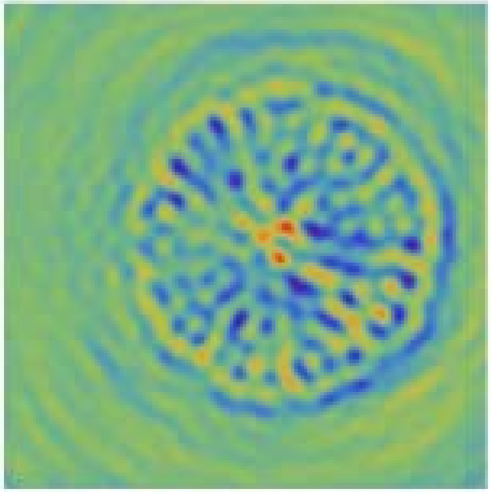


Diffusion approximation

Experimental Reconstructions: Lemon Slice



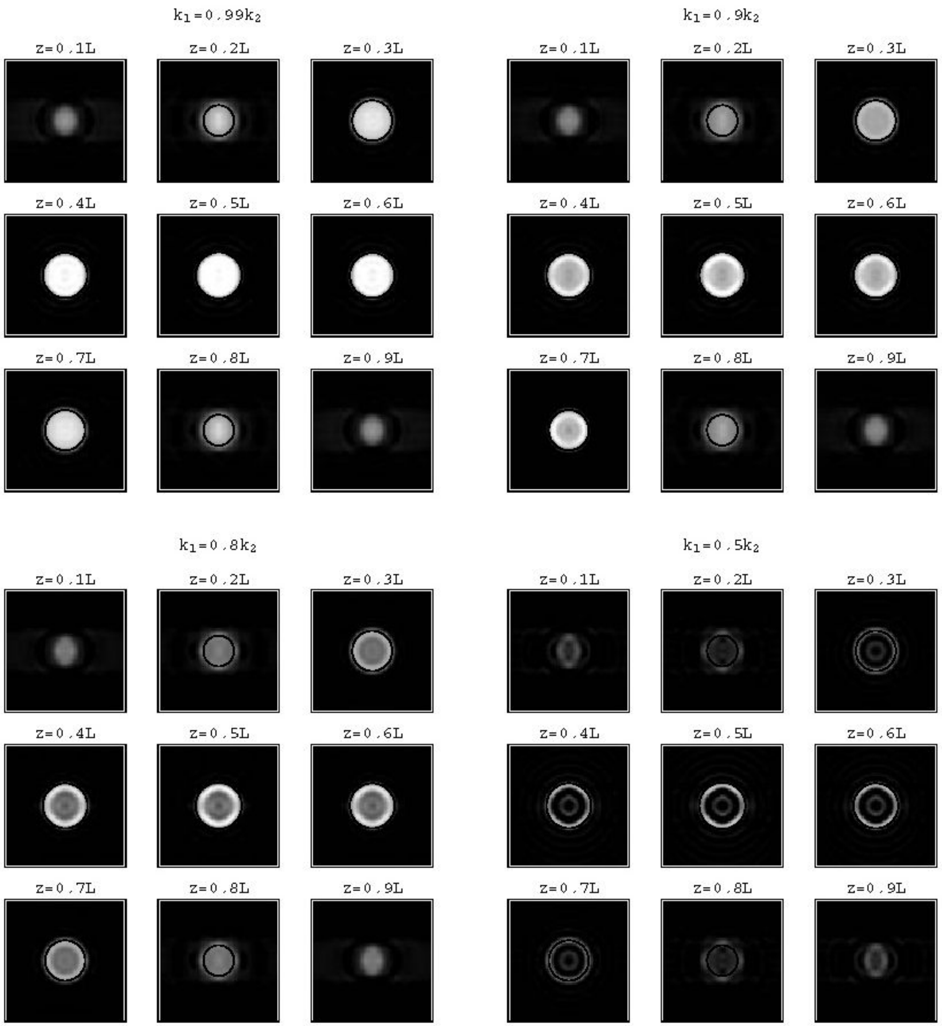
RTE



DE

Major problem: The phase function is not known and may vary.

4. NONLINEAR INVERSE PROBLEM



Why is the Inverse Problem Nonlinear?

- It's the multiple scattering.
- There is no superposition principle for the target. The field scattered from two particles in proximity to each other IS NOT equal to the sum of fields scattered by each particle separately (due to the effect of multiple scattering!)

Mathematically:

$$G(\mathbf{r}_d, \mathbf{r}_s) = G_0(\mathbf{r}_d, \mathbf{r}_s) + \int_V G_0(\mathbf{r}_d, \mathbf{r}) V(\mathbf{r}) G[V(\mathbf{r})](\mathbf{r}, \mathbf{r}_s)\, \mathrm{d}^3 r$$

The Green's function depends on the contrast (potential, interaction,etc.) that we want to reconstruct.

Formulation of the Inverse Problem in Terms of the T-matrix

$$G(\mathbf{r}_d, \mathbf{r}_s) = G_0(\mathbf{r}_d, \mathbf{r}_s) + \int_V G_0(\mathbf{r}_d, \mathbf{r}) V(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_s) d^3 r$$

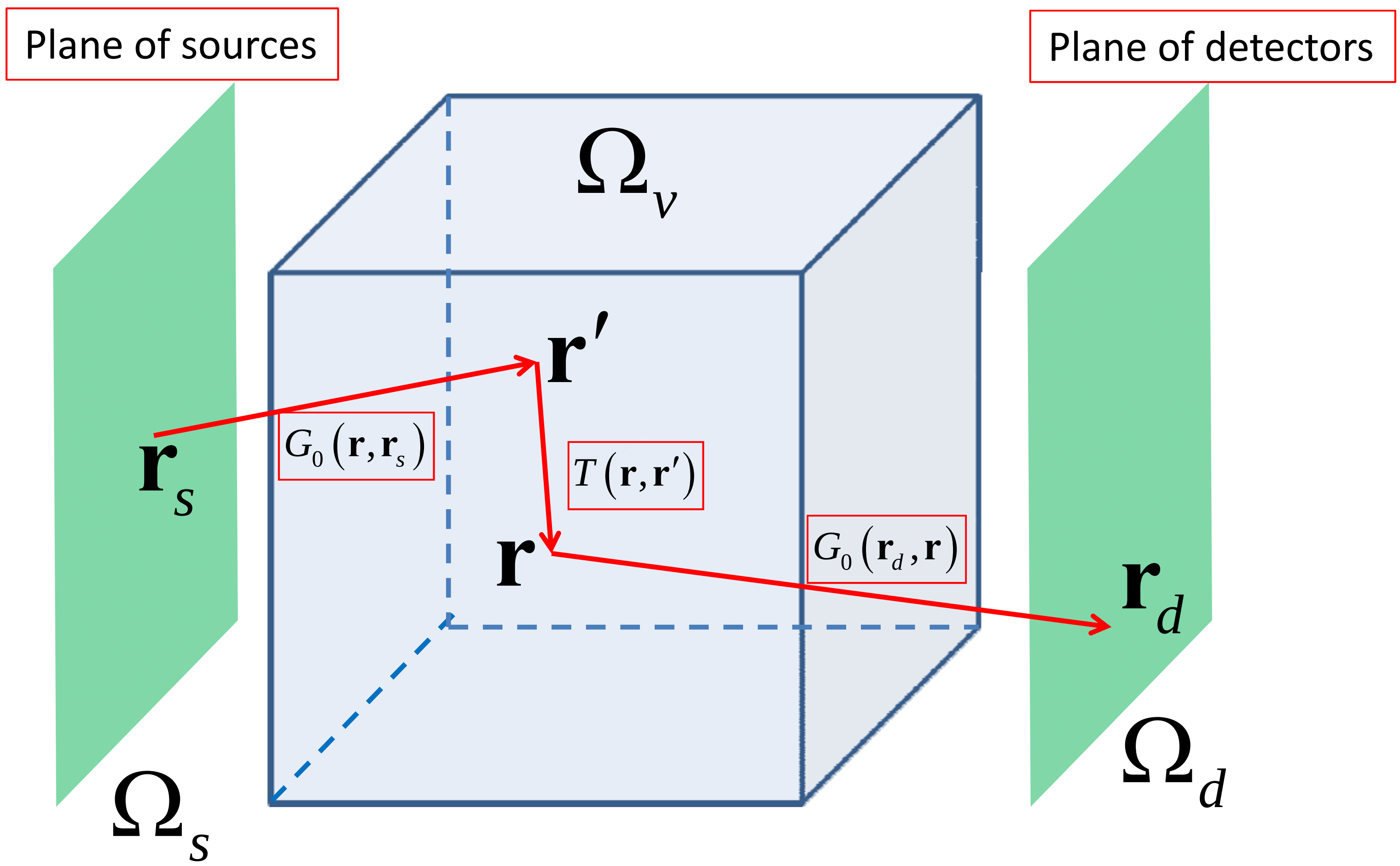
$$G = G_0 + G_0 V G \quad \Rightarrow \quad G = (1 - G_0 V)^{-1} G_0$$

$$G = G_0 + G_0 T G_0$$

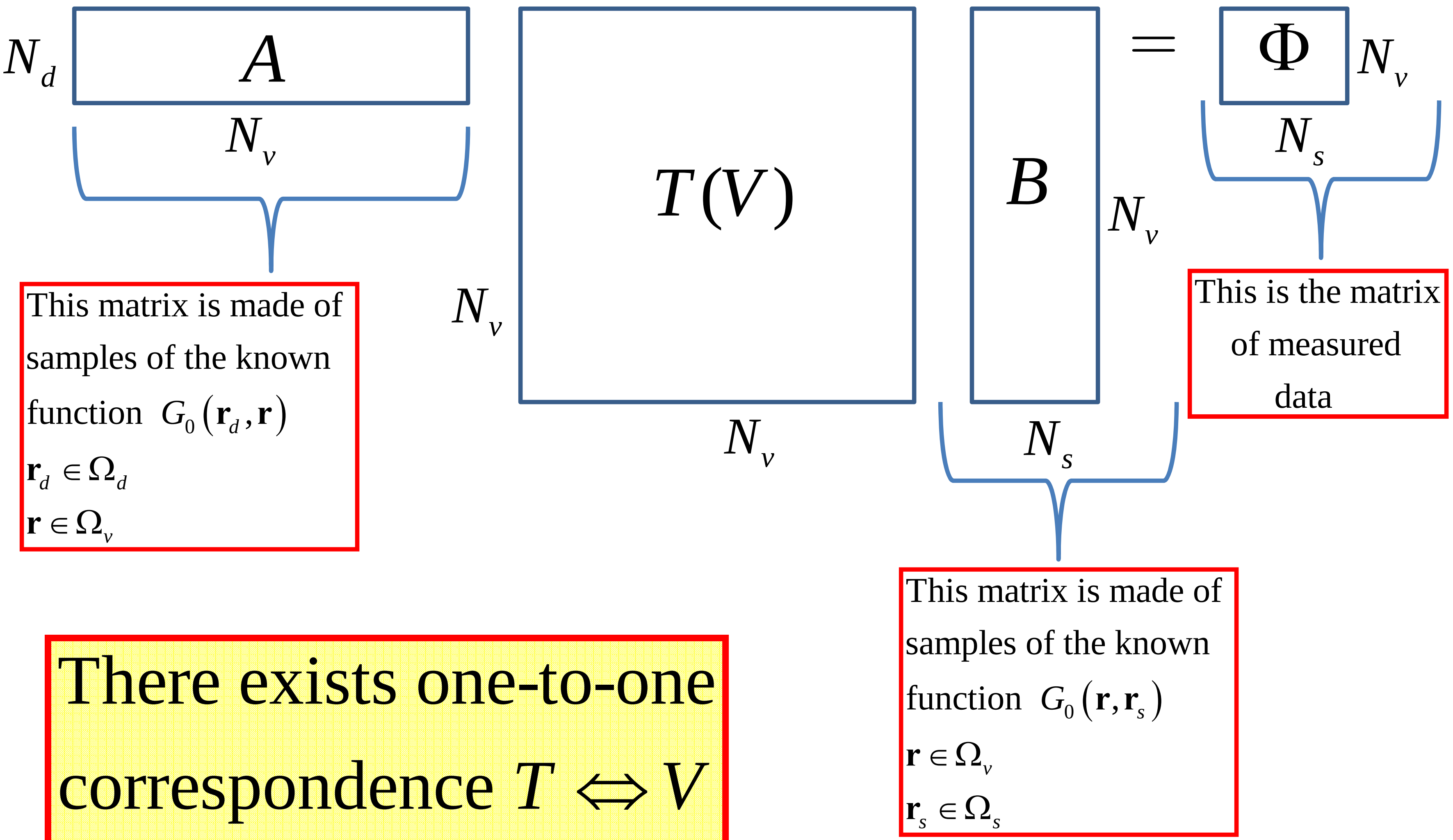
$$T G_0 = V G \quad \Rightarrow \quad T = V(1 - G_0 V)^{-1} = (1 - V G_0)^{-1} V$$

Define the matrix of data as $\Phi \equiv G - G_0$

Then $\boxed{G_0 T[V] G_0 = \Phi}$



$$G = G_0 + G_0 T G_0 \;, \quad T = (1 - V G_0)^{-1} V$$



There exists one-to-one correspondence $T \Leftrightarrow V$

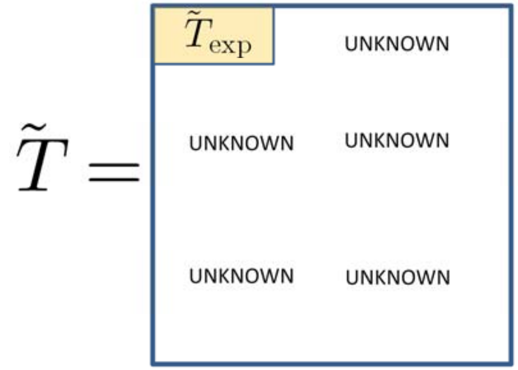
$$T = (1 - VG_0)^{-1} V$$

$$V = (1 + TG)^{-1} T$$

THE MAIN IDEA:

Find the elements of *T* that are consistent with the data and then fill the rest of the T-matrix Iteratively so that eventually it corresponds to an “almost diagonal” *V* .

Rotations of T from real space representation to singular-function representation are needed but can be done very fast numerically (not a bottleneck).



Some elements of the T-matrix in the “singular function representation” are known with certainty from data; others must be filled out from the condition that *T* corresponds to an “almost diagonal” *V* .

The number of known elements is determined by the size of the data set and by the ill-posedness of the problem but is relatively small in typical cases.

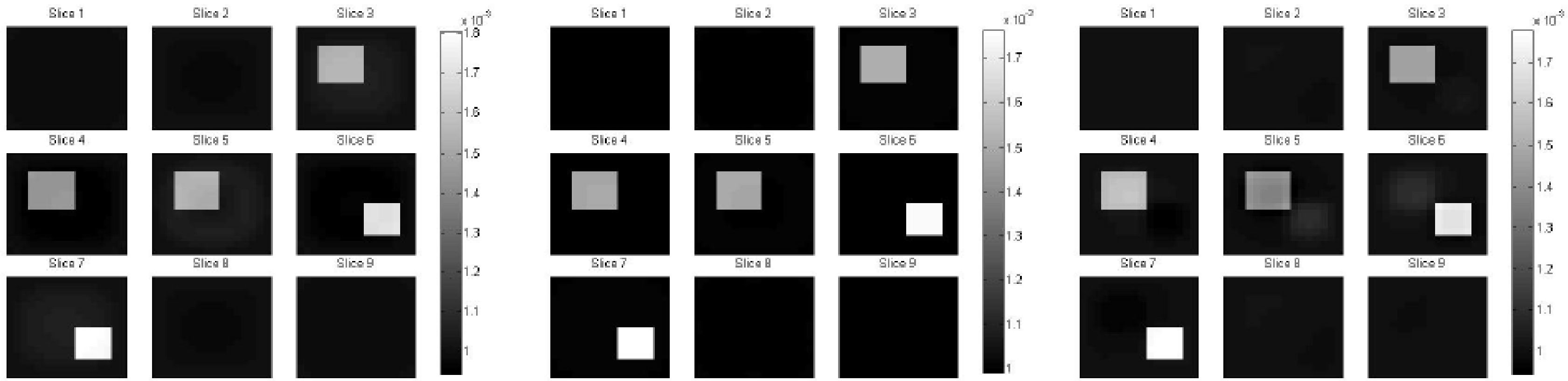
Some Useful Features

- Data are used only ONCE (to determine $\tilde{T}_{\exp}$).
- Pseudo-inversion of matrices *A* and *B* is a relatively simple task (large data sets are not a problem).
- We can adjust the iterative algorithm to solve the linear IP by simply using the map $T \leftrightarrow V$. The iterations converge to a fixed point for which there is in this case a closed-form solution, which is much faster to compute than the traditional pseudo-inverse.
- The method can be analyzed with some toy problems and has good convergence properties (converges when the IP is strongly nonlinear).
- The most time consuming operation is the transformation from T to V. The limiting factor in the computations is the number of voxels. => The method is good for strongly over-determined problems.

$$\left[\nabla^2 + k_0^2 \eta(\mathbf{r})\right] u(\mathbf{r}) = -4\pi S(\mathbf{r})$$

Sample:
Approximately $10\lambda \times 10\lambda \times 6\lambda$ rectangular box discretized into $16\times16\times9$ voxels
Number of voxels: $N_v = 2304$

Model:
$\eta = \eta_0$ everywhere except for two cubical regions:
Region (a): $3\times6\times6$ voxel box where $\eta = 1.50\,\eta_0$
Region (b): $2\times5\times5$ voxel boxwhere $\eta = 1.75\,\eta_0$



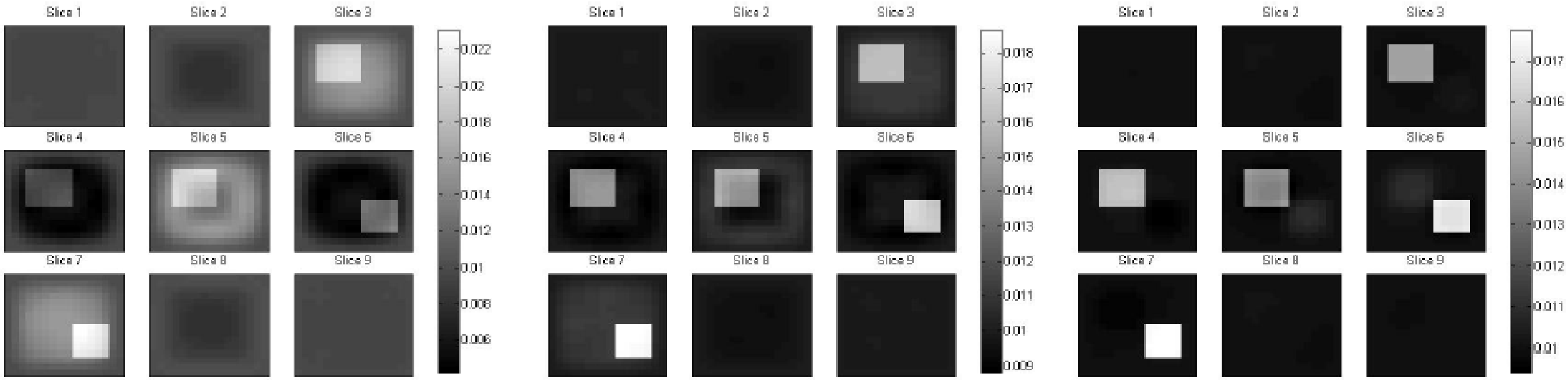
(a) Linear
$\chi = 2.60 \times 10^{-5}$

(b) Iterative Linear
$\chi = 5.39 \times 10^{-6}$

(c) Iterative Nonlinear
$\chi = 2.31 \times 10^{-5}$

Reconstructions for
$\eta_0 - 1 = 7.5 \cdot 10^{-4}$

χ is the root mean square error of the reconstruction



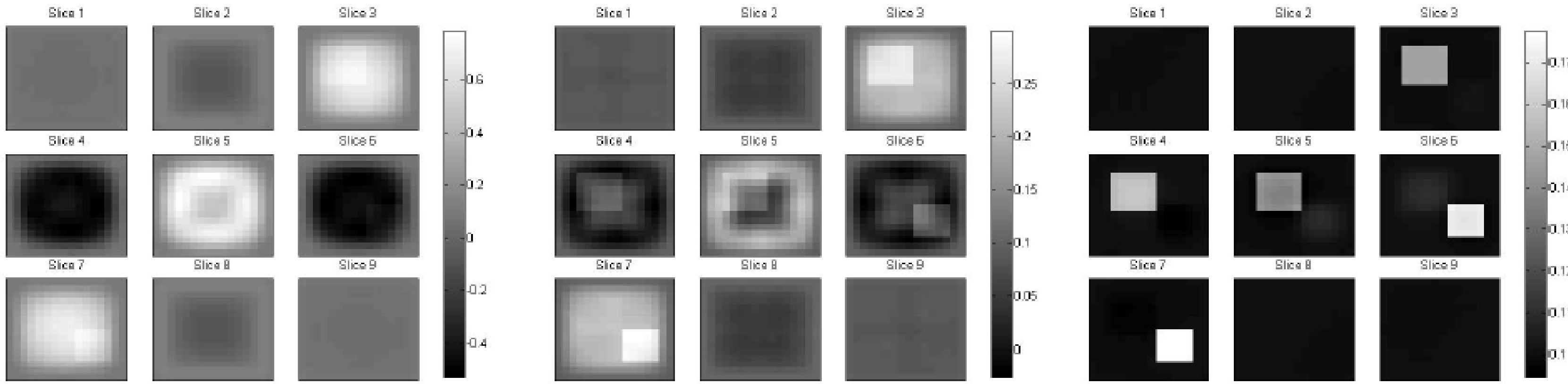
(a) Linear
$\chi = 2.60 \times 10^{-3}$

(b) Iterative Linear
$\chi = 5.38 \times 10^{-4}$

(c) Iterative Nonlinear
$\chi = 2.30 \times 10^{-4}$

Reconstructions for

$$\eta_0 - 1 = 7.5 \cdot 10^{-3}$$



(a) Linear $\chi = 0.281$

(b) Iterative Linear $\chi = 0.057$

(c) Iterative Nonlinear $\chi = 2.30 \times 10^{-3}$

Reconstructions for

$$\eta_0 - 1 = 7.5 \cdot 10^{-2}$$

$$\Delta\varphi = \frac{2\pi L_z}{\lambda}\sqrt{\eta_0 - 1} \sim \frac{\pi}{2}$$

Future Directions of Research

- Single-scattering tomography: Use of experimental data and optimization of image quality for a given total doze of radiation.
- ODT and strong scattering regime: implement-ation of the nonlinear algorithms, other fast algorithms for large experimental datasets (in particular, with amplitude-modulated sources)
- RTE and intermediate scattering regime: Wait for better data/info on the phase function.
- NONLINEAR IP: further development of the method (many things to try here)