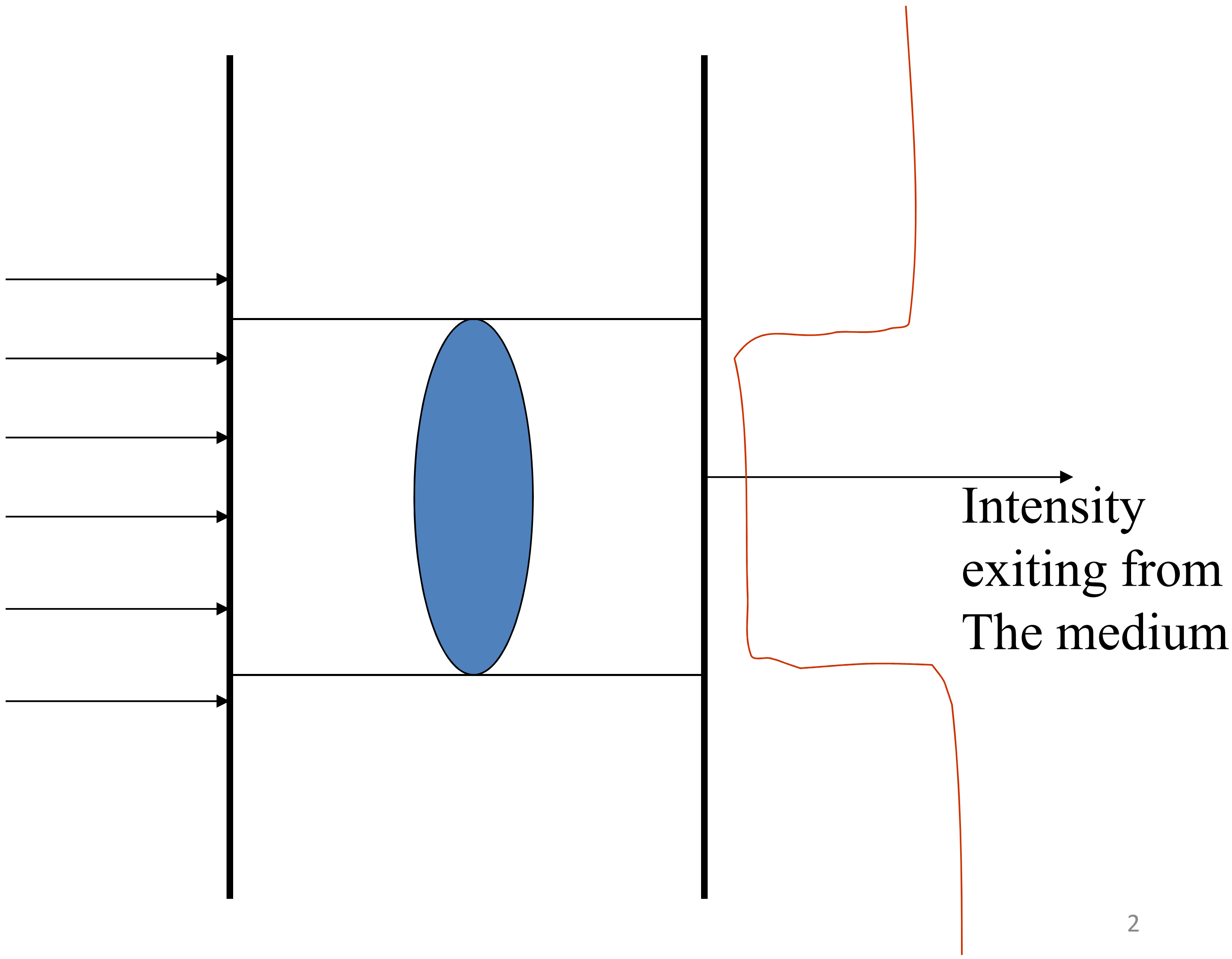
BROKEN-RAY TOMOGRAPHY

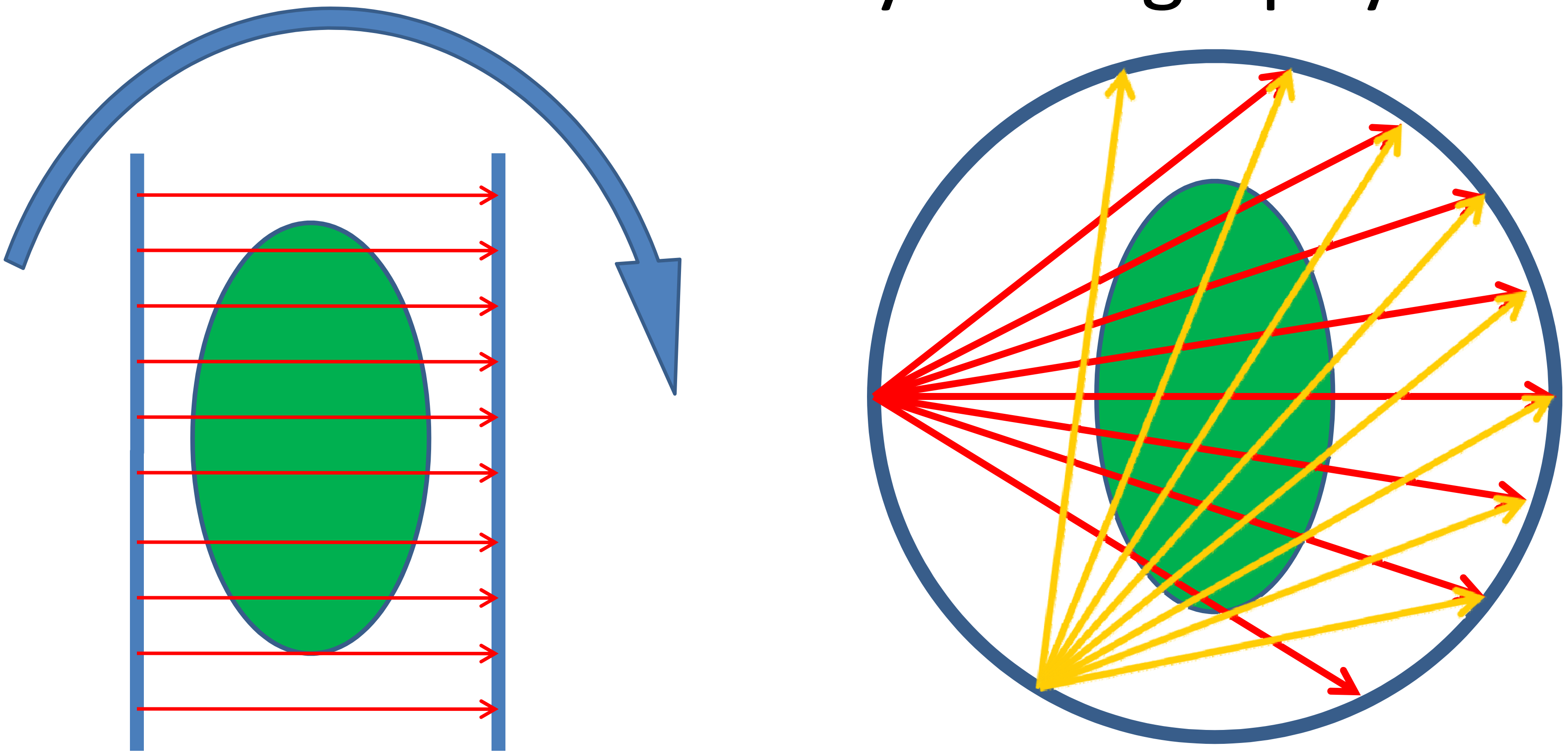
Vadim A. Markel
University of Pennsylvania

A joint project with:
Lucia Florescu (formerly at Upenn)
Fan Zhao (Upenn)
John Schotland (UMich)
Alexander Katsevich (U Central Florida)

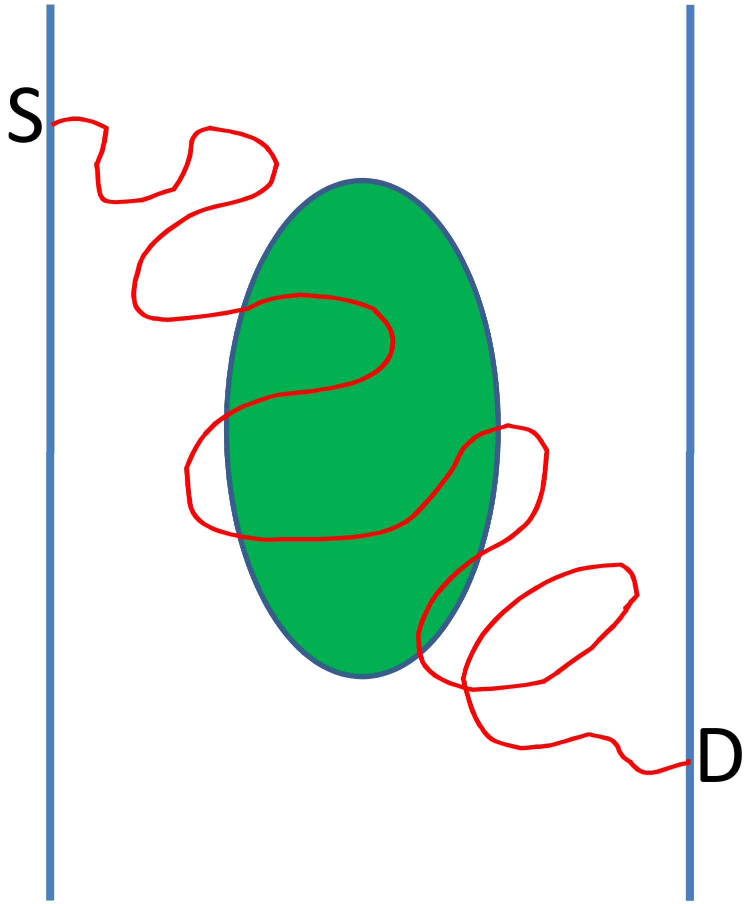
X-Ray Imaging



Zero Scattering Regime: Conventional X-Ray Tomography

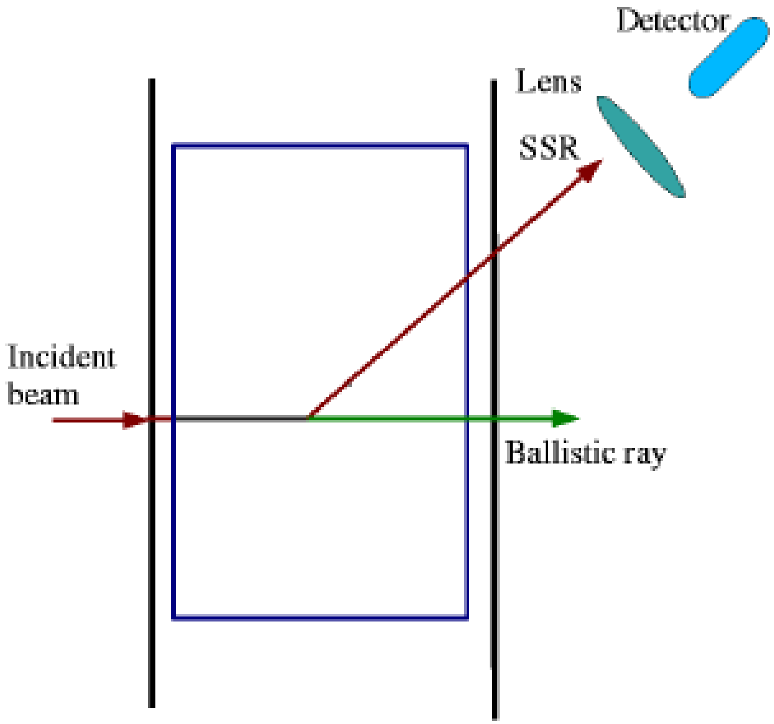


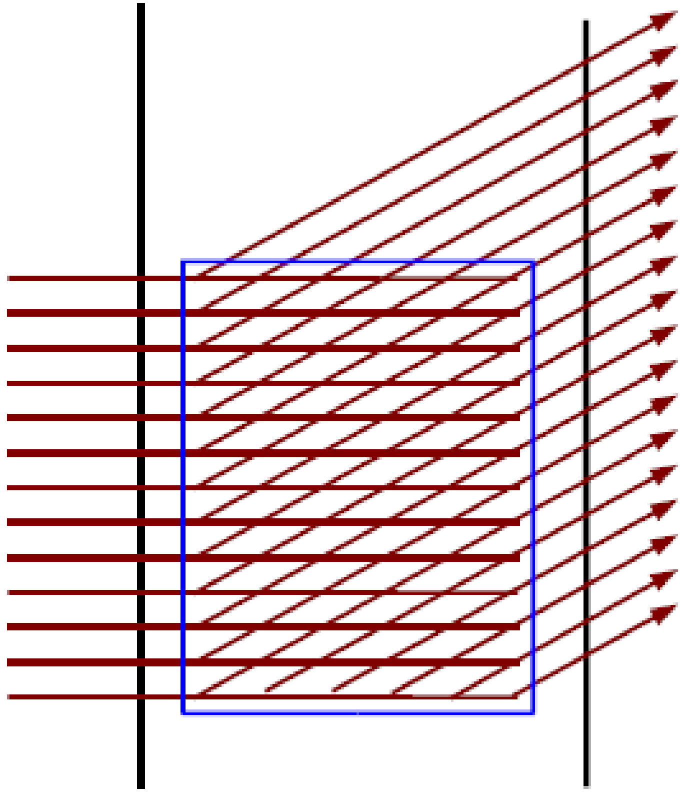
Strong Scattering Regime: Diffuse Optical Tomography



- Many source-detector pairs
- Severely ill-posed IP
- Nonlinear IP

Mesoscopic Scattering Regime: Single-Scattering Tomography





SSOT And Other Modalities

	Linear IP	IP is MILDLY Ill-posed	Single projection	Reflection geometry	Nonionizing radiation	Quantitatrive images	Reconstruc-tion of scattering and absorption
X-ray CT	Y (if *E*=const)	Y	N	N	N	Y	N
DOT	N	N	Y	Y	Y	?	Y (with time or freq.)
SSOT	Y	Y	Y	Y	Y	Y	Y (with two det. angles)

The Broken-Ray Integral Transform

(a) $\mu_s = \text{const}$ (and is known)

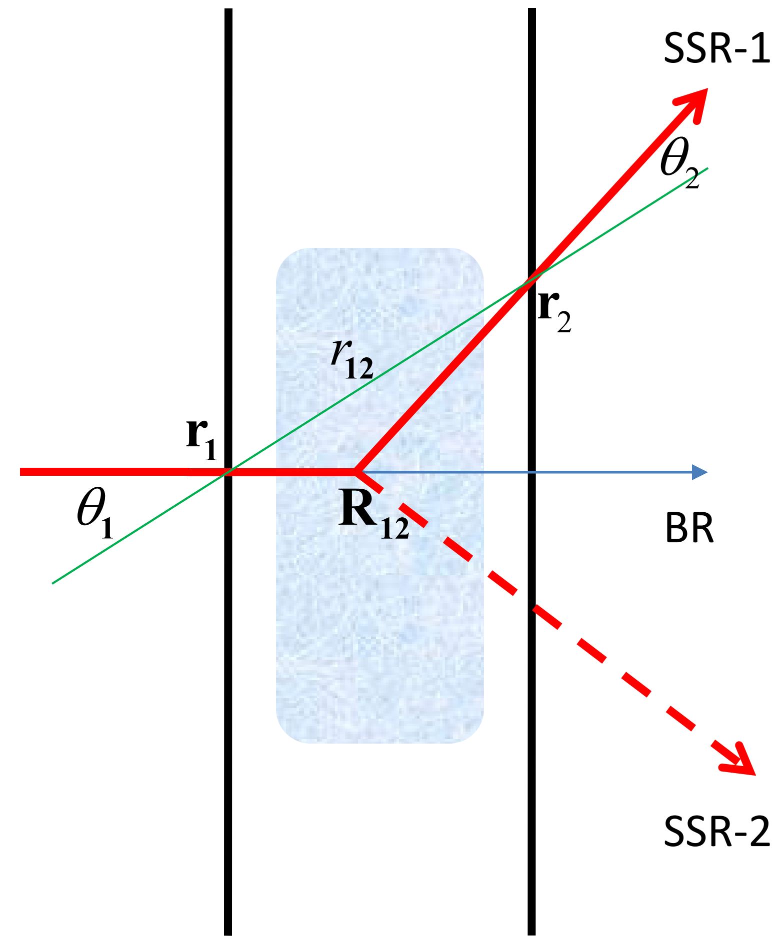
$$\int_{\text{SSR}} \mu_t[\mathbf{r}(\ell)]d\ell = \phi(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}_2, \hat{\mathbf{s}}_2)$$

(b) $\mu_s \neq \text{const}$ (and is unknown)

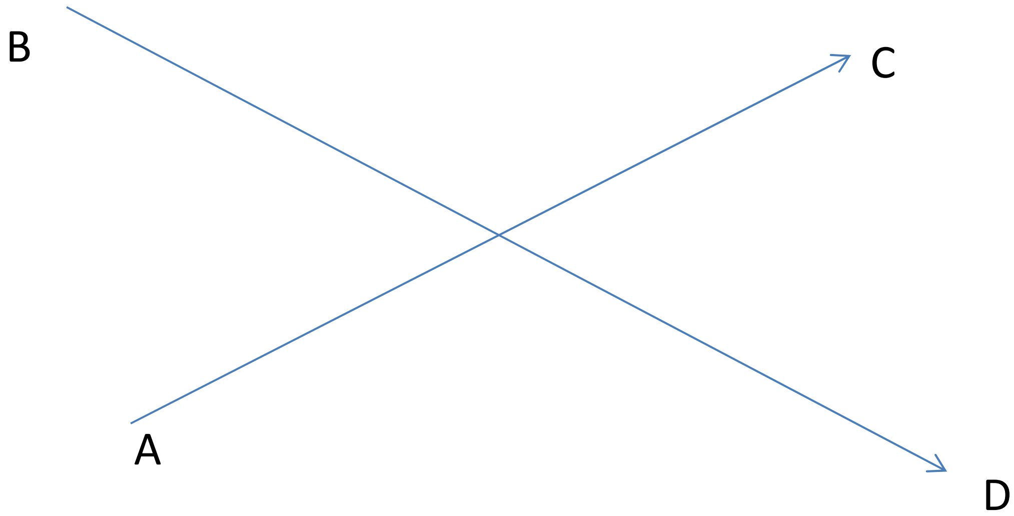
$$\int_{\text{SSR}} \mu_t[\mathbf{r}(\ell)]d\ell - \ln\left[\frac{\mu_s(\mathbf{R}_{12})}{\langle \mu_s \rangle}\right] = \phi(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}_2, \hat{\mathbf{s}}_2)$$

The measurable data function:

$$\phi(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}_2, \hat{\mathbf{s}}_2) = -\ln\left[\frac{r_{12}\sin\theta_1 \sin\theta_2}{\langle \mu_s \rangle A(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2)} \frac{I_{\text{measured}}}{I_{\text{incident}}}\right]$$



Broken rays: You can go two ways



1) A→C
   B→D

2) A→D
   B→C

Outline:

1) Numerical test (40x40 rays per slice), full RTE forward solver, no inverse crime
2) Generalized filtered back-projection formula
3) Inverse crime simulations based on this formula (many rays)

PART 1: Numerical simulations: Data from the RTE (no inverse crime)

- Forward model based on the RTE
- Isotropic scattering
- FULL ACCOUNT OF MULTIPLE SCATTERING
- BOUNDARY CONDITIONS SATISFIED EXACTLY
- 3D integral equation for density discretized on a rectangular grid
- Direct inversion of a well-posed square matrix
- Mathematical details on next page…

$$[\hat{\mathbf{s}}\cdot\nabla + \underbrace{\mu_a(\mathbf{r}) + \mu_s(\mathbf{r})}_{\mu_t(\mathbf{r})}]I(\mathbf{r},\hat{\mathbf{s}}) = \mu_s(\mathbf{r})\int \underbrace{\frac{1}{4\pi}}_{A(\hat{\mathbf{s}},\hat{\mathbf{s}}')=\frac{1}{4\pi}=\text{const}} I(\mathbf{r},\hat{\mathbf{s}}')d^2\hat{\mathbf{s}}'$$

$$u(\mathbf{r}) = \int I(\mathbf{r},\hat{\mathbf{s}})d^2\hat{\mathbf{s}}$$

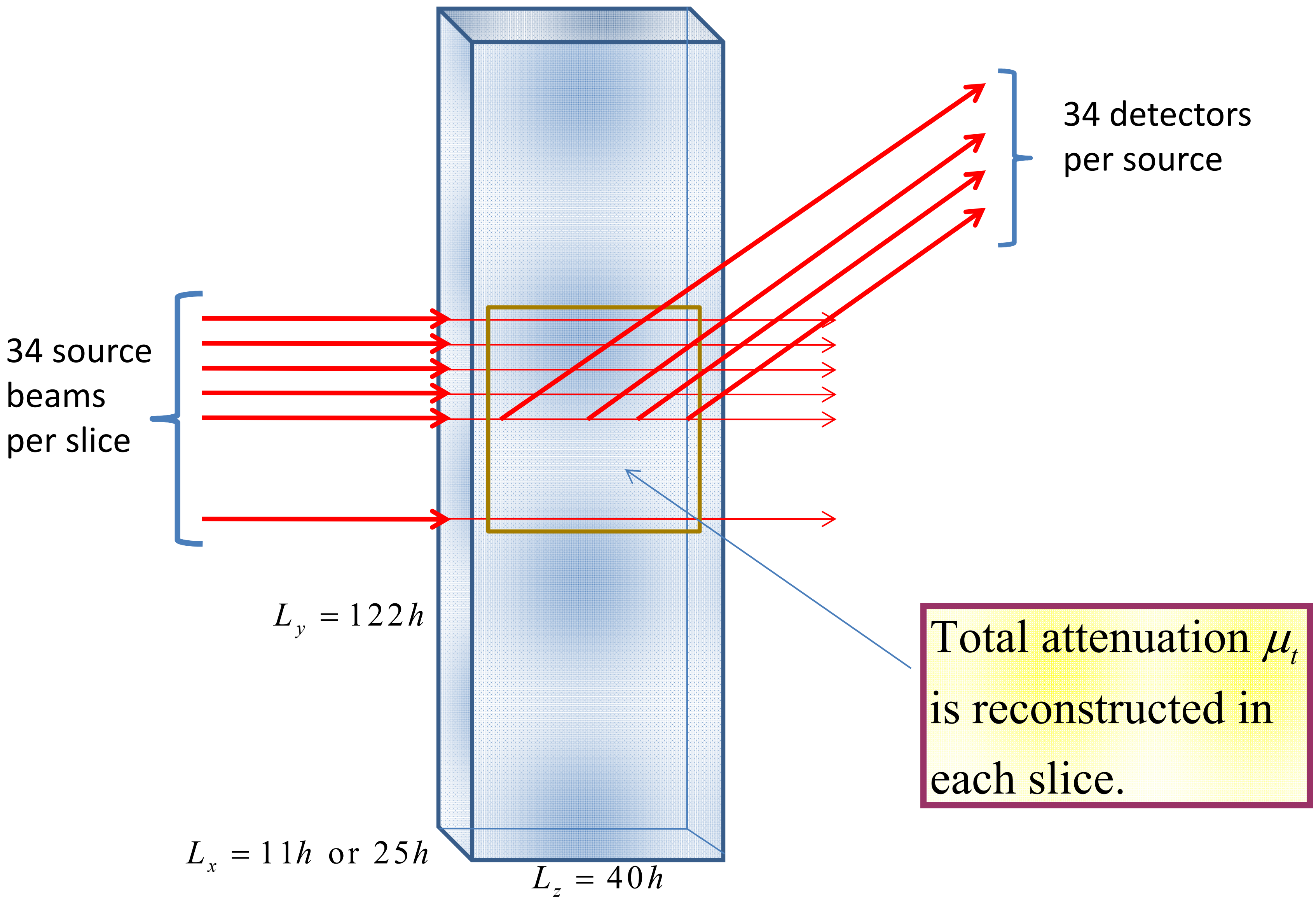
$$u(\mathbf{r}) = u_b(\mathbf{r}) + \int g_b(\mathbf{r},\mathbf{r}')\frac{\mu_s(\mathbf{r}')}{4\pi}u(\mathbf{r}')d^3r'$$

$$I(\mathbf{r},\hat{\mathbf{s}}) = I_b(\mathbf{r},\hat{\mathbf{s}}) + \int G_b(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')\frac{\mu_s(\mathbf{r}')}{4\pi}u(\mathbf{r}')d^3r'd^2\hat{\mathbf{s}}'$$

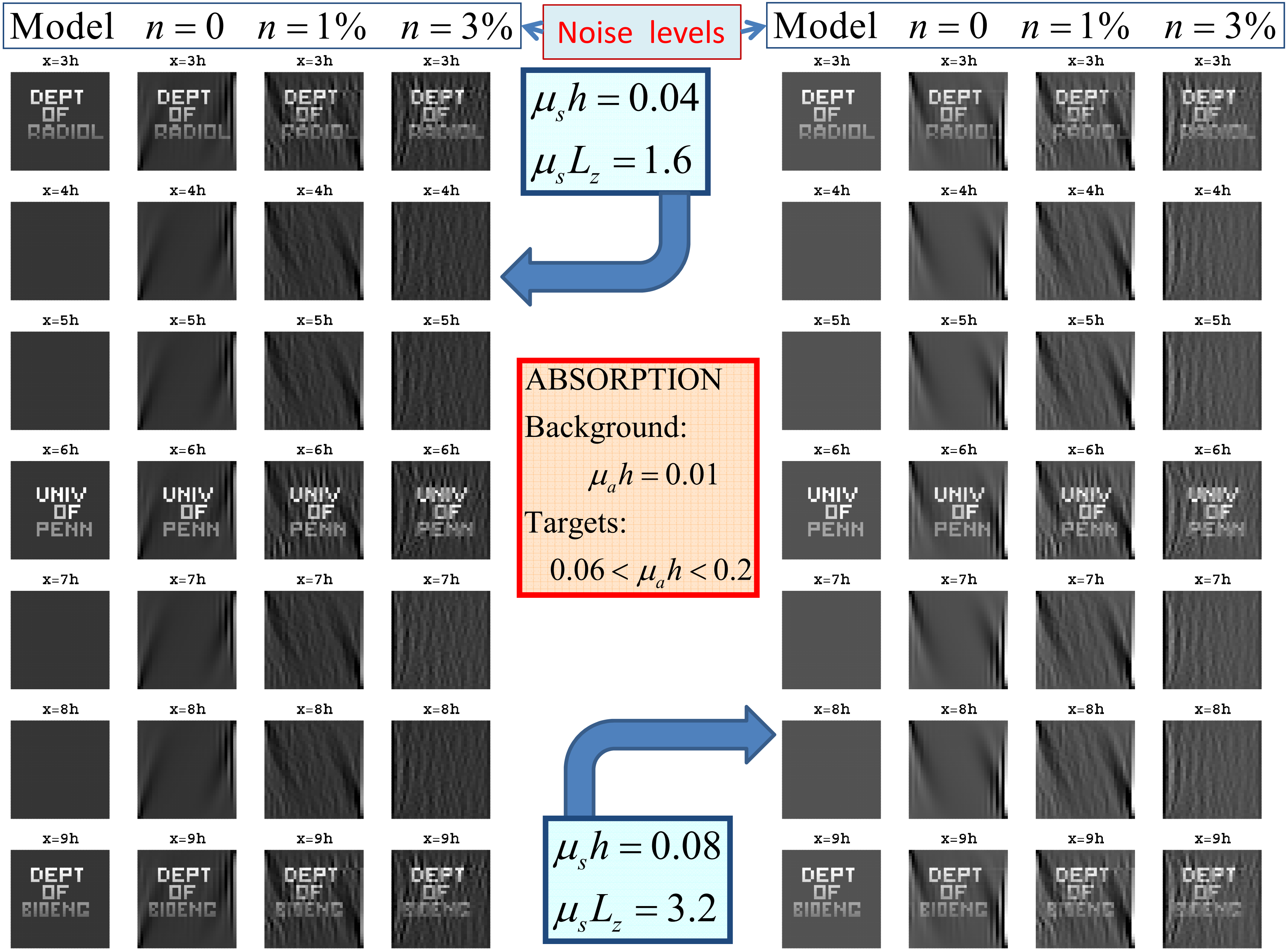
$$G_b(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}') = g_b(\mathbf{r},\mathbf{r}')\delta\left(\hat{\mathbf{u}}(\mathbf{r}-\mathbf{r}')-\hat{\mathbf{s}}'\right)\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}')$$

$$g_b(\mathbf{r},\mathbf{r}') = \int G_b(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}')d^2\hat{\mathbf{s}}d^2\hat{\mathbf{s}}' =$$

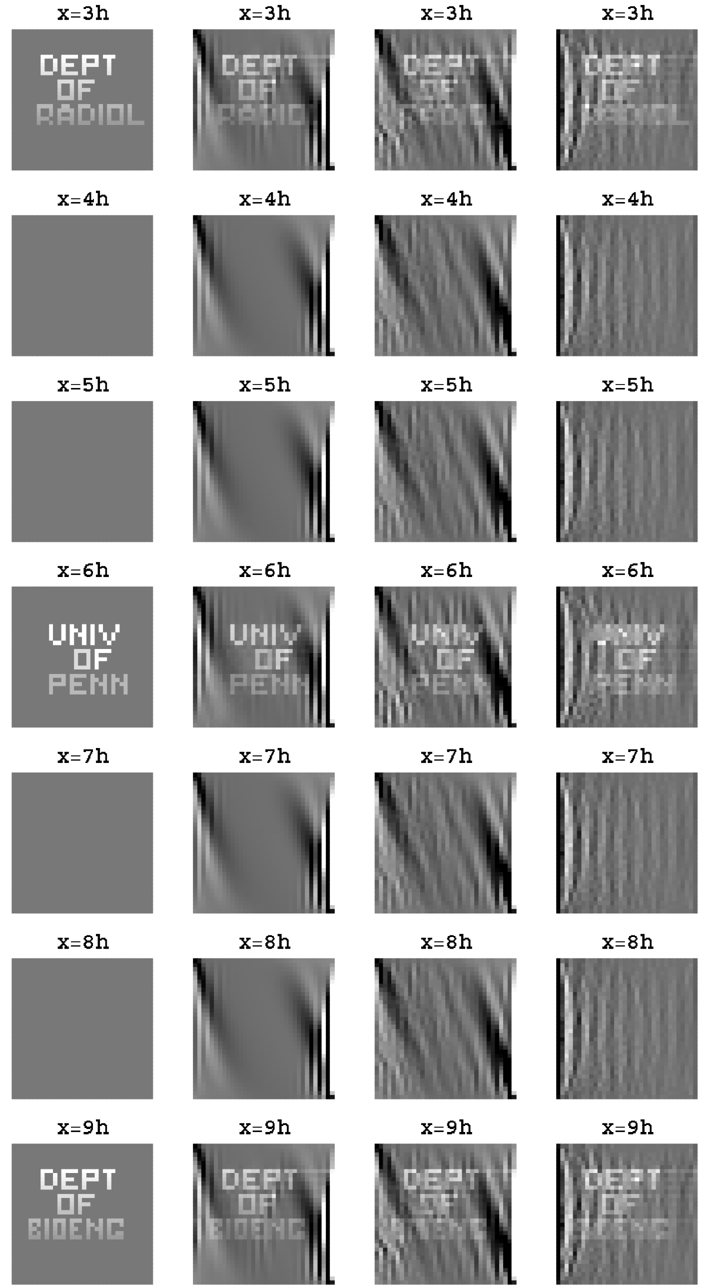
$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\exp\left[-\int_0^{|\mathbf{r}-\mathbf{r}'|}\mu_t\left(\mathbf{r}'+\ell\hat{\mathbf{u}}(\mathbf{r}-\mathbf{r}')\right)d\ell\right]$$



34 detectors per source
34 source beams per slice
$L_y = 122h$
$L_x = 11h$ or $25h$
$L_z = 40h$
Total attenuation μ_t is reconstructed in each slice.


Model $n=0$ $n=1\%$ $n=3\%$
Noise levels
Model $n=0$ $n=1\%$ $n=3\%$
$\mu_s h = 0.04$
$\mu_s L_z = 1.6$
ABSORPTION
Background:
$\mu_a h = 0.01$
Targets:
$0.06 < \mu_a h < 0.2$
$\mu_s h = 0.08$
$\mu_s L_z = 3.2$
x=3h
x=4h
x=5h
x=6h
x=7h
x=8h
x=9h
DEPT OF RADIOL
UNIV OF PENN
DEPT OF BIOENG

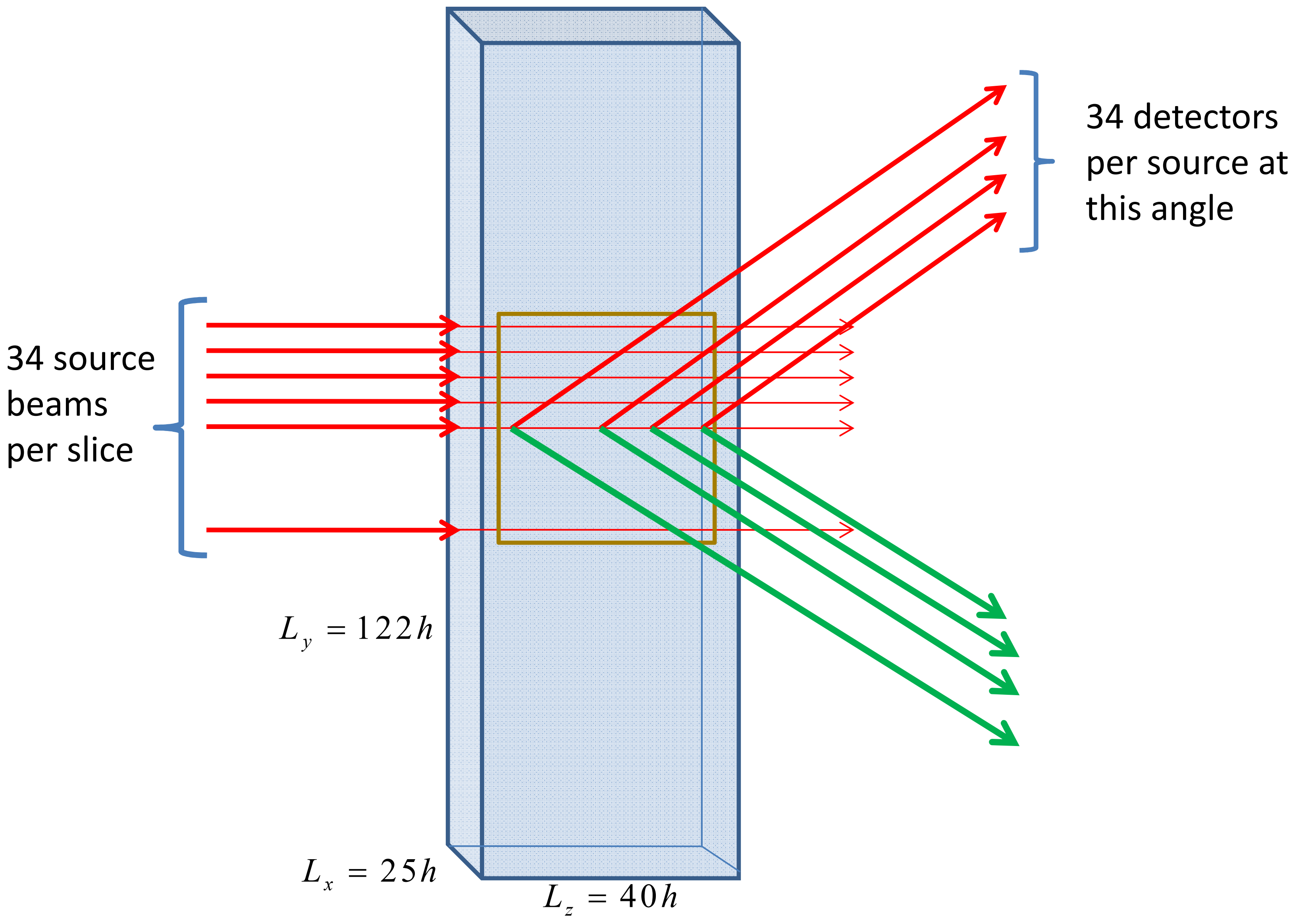
Model $n = 0$ $n = 1\%$ $n = 3\%$



$\mu_s = 0.16h^{-1}$

$\mu_s L_z = 6.4$

Reconstruction of scattering+absorption in a thicker sample: Lx=25h



Simultaneous reconstruction of absorption and scattering

SCATTERING

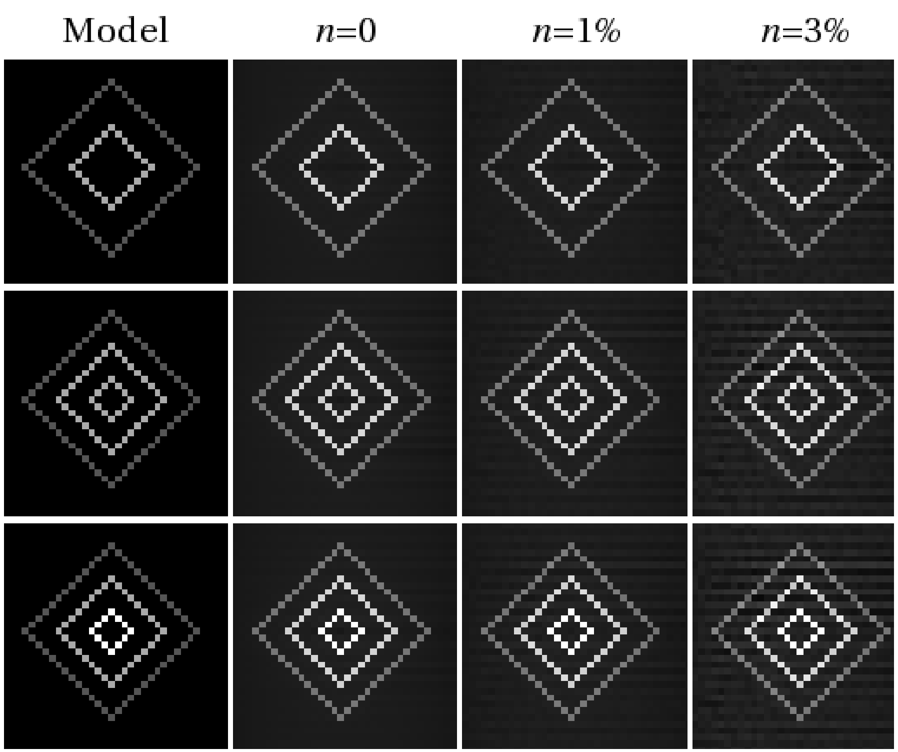
Background: $\overline{\mu}_s L_z = 1.6$

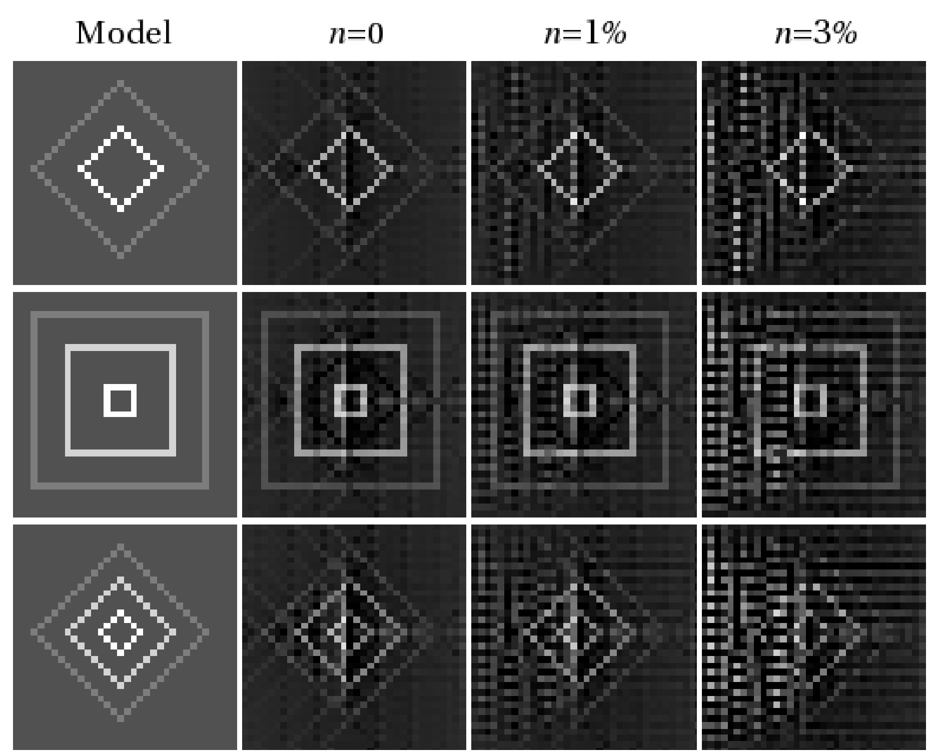
Targets: $1.33\overline{\mu}_s \le \mu_s \le 2\overline{\mu}_s$

ABSORPTION

Background: $\overline{\mu}_a = 0.1\overline{\mu}_s$

Targets: $2\overline{\mu}_a \le \mu_a \le 5\overline{\mu}_a$





Inhomogeneities with stronger scattering

SCATTERING

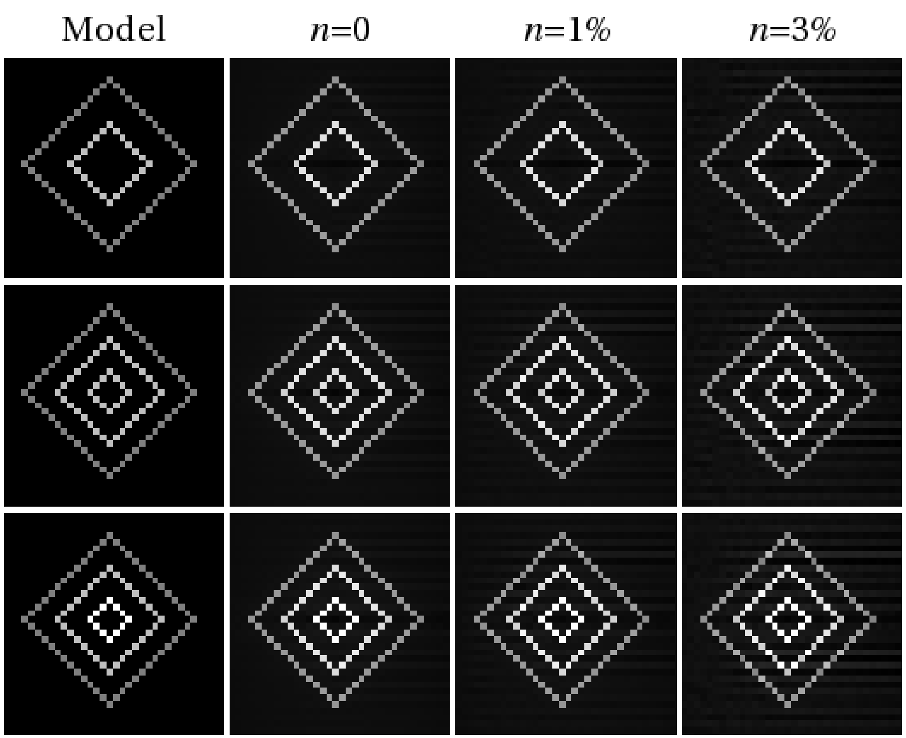
Background: $\bar{\mu}_s L_z = 1.6$

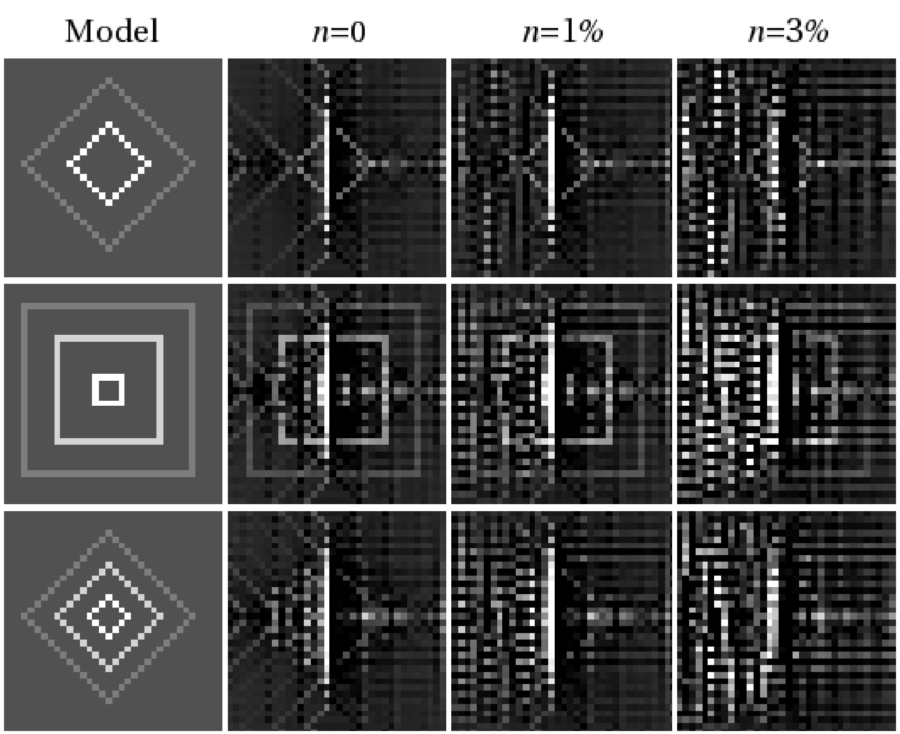
Targets: $2\bar{\mu}_s \le \mu_s \le 3\bar{\mu}_s$

ABSORPTION

Background: $\bar{\mu}_a = 0.1\bar{\mu}_s$

Targets : $2\bar{\mu}_a \le \mu_a \le 5\bar{\mu}_a$





Stronger absorption (overall)

SCATTERING

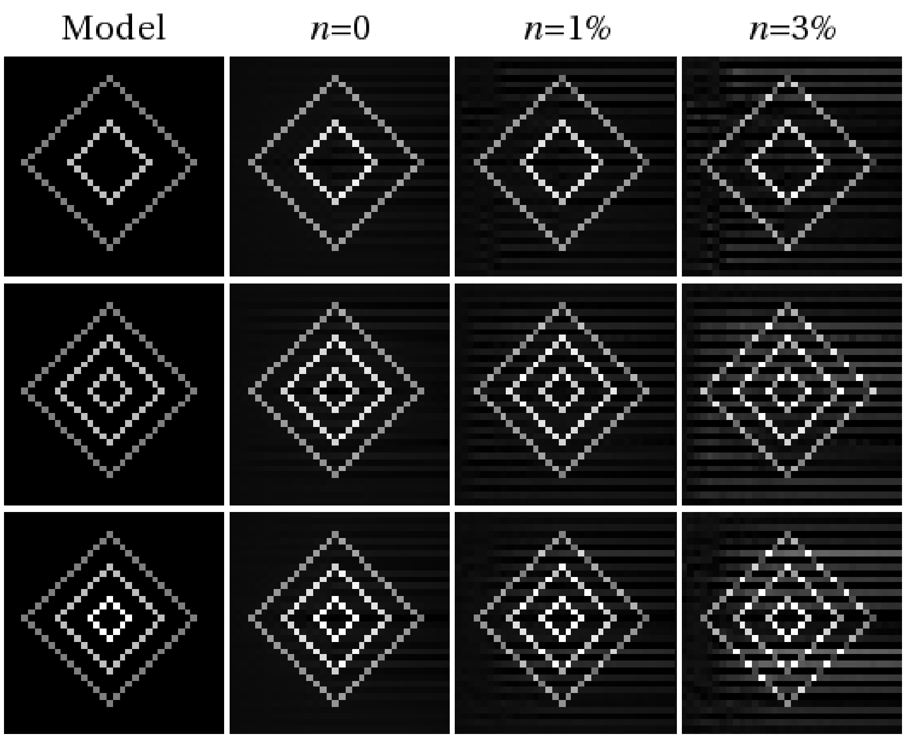
Background: $\overline{\mu}_s L_z = 1.6$

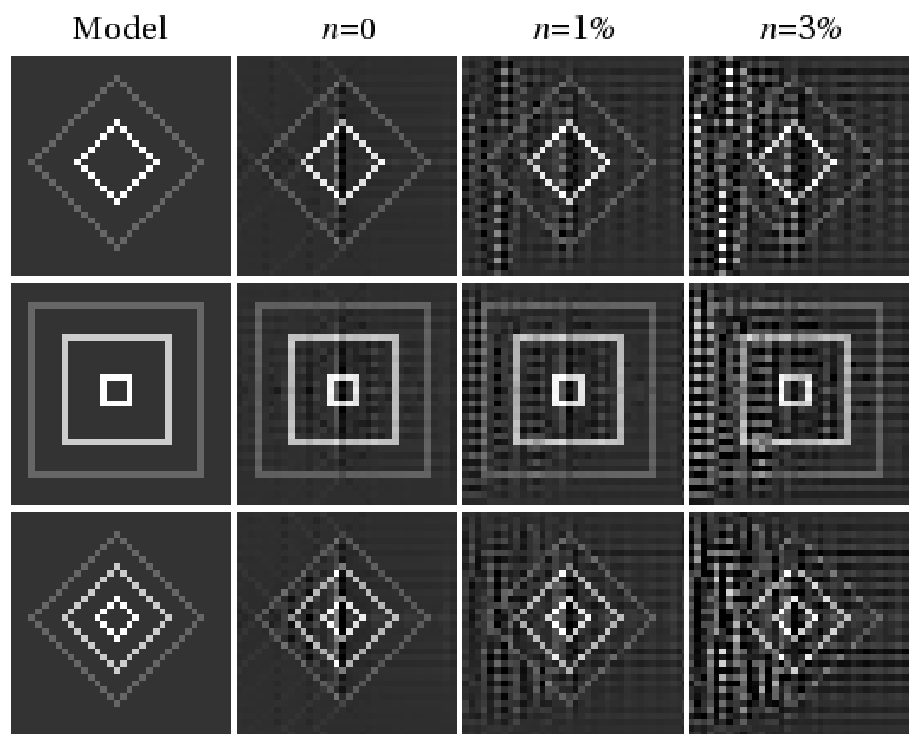
Targets: $2\overline{\mu}_s \le \mu_s \le 3\overline{\mu}_s$

ABSORPTION

Background: $\overline{\mu}_a = \overline{\mu}_s$

Targets : $2\overline{\mu}_a \le \mu_a \le 5\overline{\mu}_a$





Same as before, but larger optical depth

SCATTERING

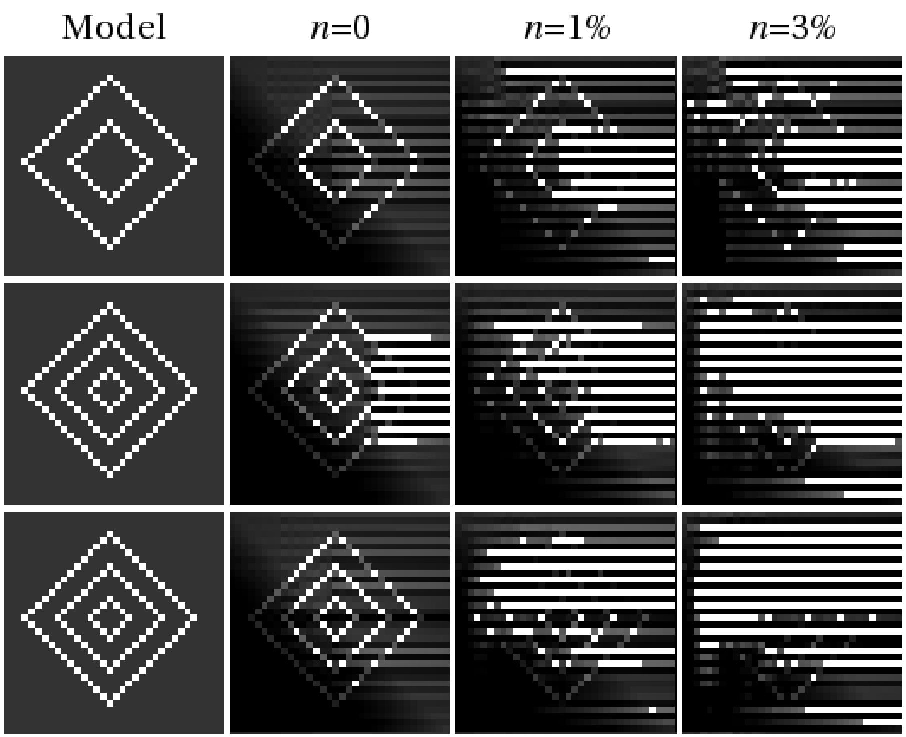
Background: $\overline{\mu}_s L_z = 3.2$

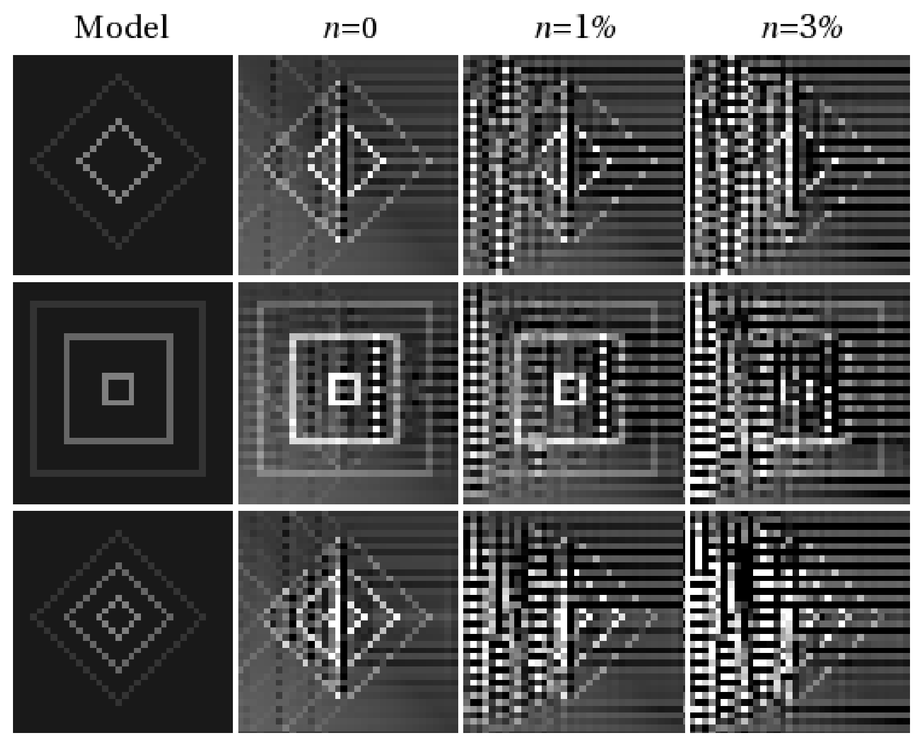
Targets: $2\overline{\mu}_s \le \mu_s \le 3\overline{\mu}_s$

ABSORPTION

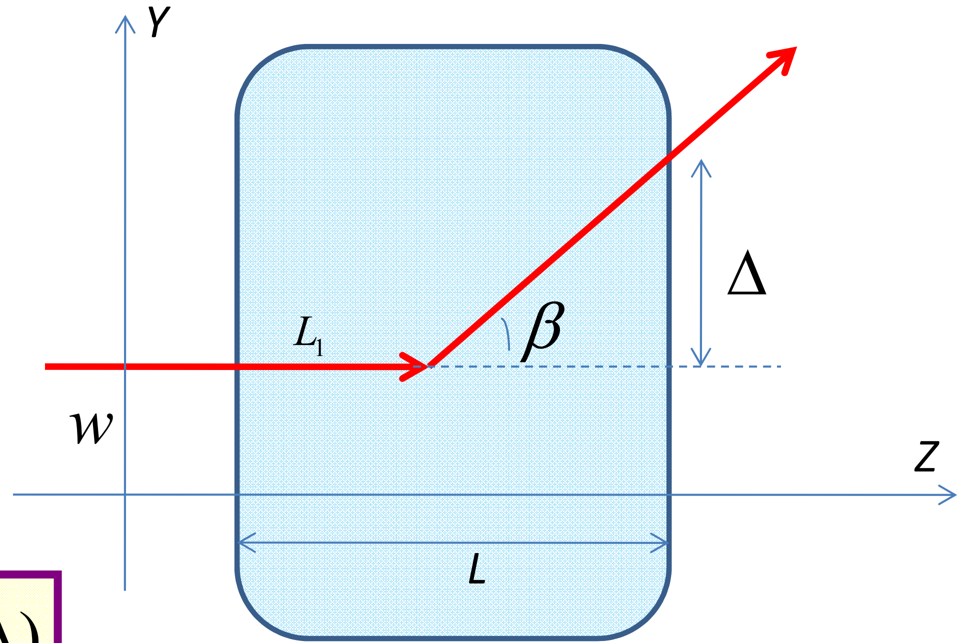
Background: $\overline{\mu}_a = \overline{\mu}_s$

Targets: $2\overline{\mu}_a \le \mu_a \le 5\overline{\mu}_a$





PART 2: FBF derivation



$$\int_{\mathrm{SSR}(w,\Delta)} \mu_t[\boldsymbol{r}(\ell)]d\ell = \phi(w,\Delta)$$

$$\boldsymbol{r} = (y, z)$$

$$y(\ell) = w + \eta(\Delta, \ell)$$

$$z(\ell) = \zeta(\Delta, \ell)$$

$$\eta(\Delta,\ell)=\begin{cases}0, & \ell<L_1(\Delta)\\ \ell-L_1(\Delta), & L_1(\Delta)<\ell<L_1(\Delta)+L_2(\Delta)\end{cases}$$

$$\zeta(\Delta,\ell)=\begin{cases}\ell, & \ell<L_1(\Delta)\\ L_1(\Delta)+\left[\ell-L_1(\Delta)\right]\cos(\beta), & L_1(\Delta)<\ell<L_1(\Delta)+L_2(\Delta)\end{cases}$$

$$L_1(\Delta)=L-\Delta\,\mathrm{ctg}(\beta)\quad,\qquad L_2(\Delta)=\Delta/\sin(\beta)$$

The ray langth (within the medium): $L_1(\Delta)+L_2(\Delta)$

Fourier slice theorem

$$\tilde{\phi}(k,\Delta) = \int_0^{L_1(\Delta)+L_2(\Delta)} e^{ik\eta(\Delta,\ell)} \tilde{\mu}_t[k,\zeta(\Delta,\ell)]d\ell$$

$$\tilde{\phi}(k,\Delta) = \int_0^{L_1(\Delta)} \tilde{\mu}_t(k,\ell)d\ell + \frac{e^{ikL_1(\Delta)\operatorname{tg}\beta}}{\cos\beta} \int_{L_1(\Delta)}^{L} \tilde{\mu}_t(k,\ell)e^{-ik\ell\operatorname{tg}\beta}d\ell$$

Define new variables: $q = k\operatorname{tg}\beta \ ; \ c = \cos\beta$

$$f(z) = \tilde{\mu}_t(q\operatorname{ctg}\beta, z) \ ;$$

$$F(z) = \tilde{\phi}\left[q\operatorname{ctg}\beta, (L-z)\operatorname{tg}\beta\right]$$

$$\int_0^z f(\ell)d\ell + \frac{1}{c}e^{iqz}\int_z^L e^{-iq\ell} f(\ell)d\ell = F(z)$$

$$c = \cos\beta \ , \qquad\qquad 0 < z < L$$

The inverse solution:

$$f_i(z) = -\kappa\left[G(z) - i\kappa q e^{-i\kappa q z}\int_0^z e^{i\kappa q\ell} G(\ell)d\ell \right]$$

$$\kappa = \frac{c}{1-c} \ ; \ \ G(z) = \left(\frac{d}{dz} - iq\right)F(z)$$

Putting everything together…

$$\tilde{\mu}_t(k,z) = \sigma \left\{ H(k,z) - ik\sigma e^{-ik\sigma z} \int_0^z e^{i\sigma k\ell} H(k,\ell) dl \right\}$$

$$\sigma = \mathrm{ctg}\left(\frac{\beta}{2}\right) ;$$

$$H(k,z) = \left(\frac{\partial}{\partial\Delta} + ik\right)\tilde{\phi}(k,\Delta)\Big|_{\Delta=(L-z)\,\mathrm{tg}\,\beta}$$

The real-space inversion formula

$$\mu_t(y,z) = \int_{-\infty}^{\infty} \tilde{\mu}_t(k,z) e^{-iky} \frac{dk}{2\pi}$$

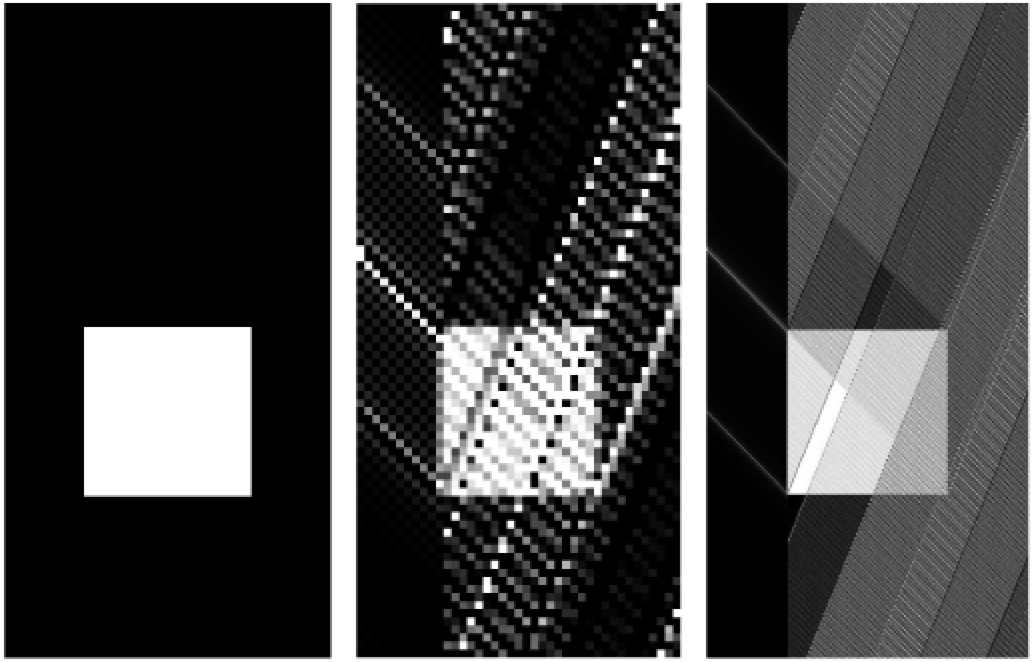
$$\mu_t(y,z) = \sigma \left\{ \left(\frac{\partial}{\partial \Delta} - \frac{\partial}{\partial y} \right) \phi(y,\Delta) + \tau \frac{\partial}{\partial y} \left[\phi(y + \sigma z, L \operatorname{tg} \beta) - \phi(y,\Delta) \right] \right.$$
$$\left. \left. - (1+\tau) \frac{\partial}{\partial y} \int_{\Delta}^{L \operatorname{tg} \beta} \phi(y + \tau(\ell - \Delta), \ell) \, d\ell \right\} \right|_{\Delta = (L-z) \operatorname{tg} \beta}$$

$$\sigma = \operatorname{ctg}\left(\frac{\beta}{2} \right) ; \quad \tau = \operatorname{ctg}\left(\frac{\beta}{2} \right) / \operatorname{tg}(\beta)$$

PART 3: Inverse-crime simulations

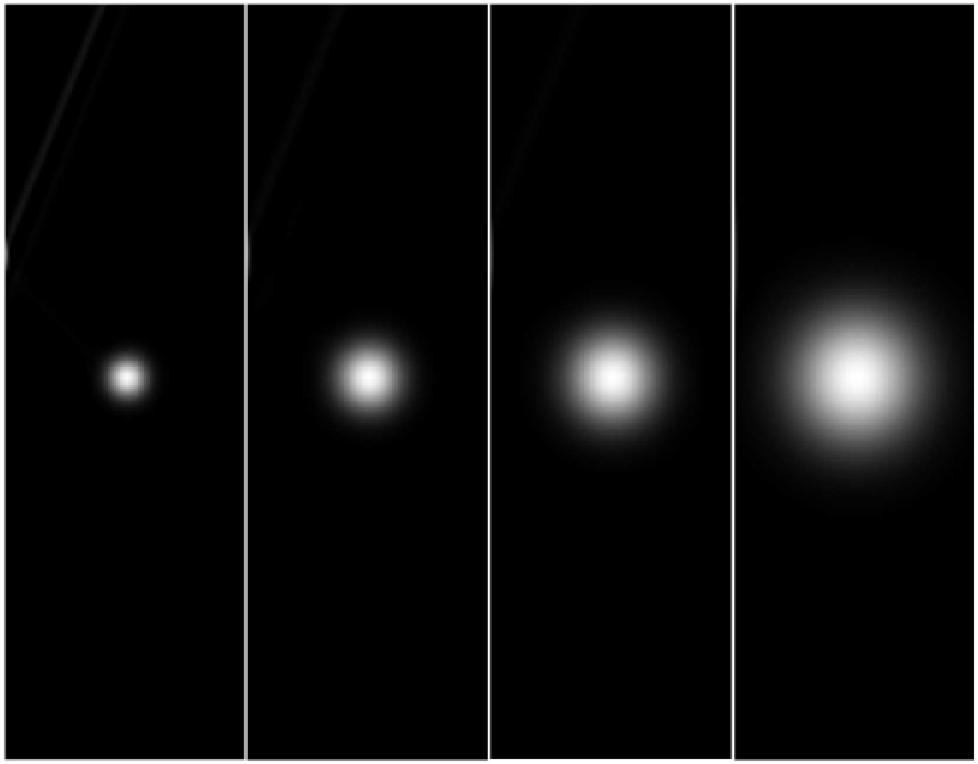
Reconstruction using the Fourier-space formula

$$\beta = \frac{\pi}{4}$$



Model L/h=40 L/h=400

Reconstruction of a Gaussian

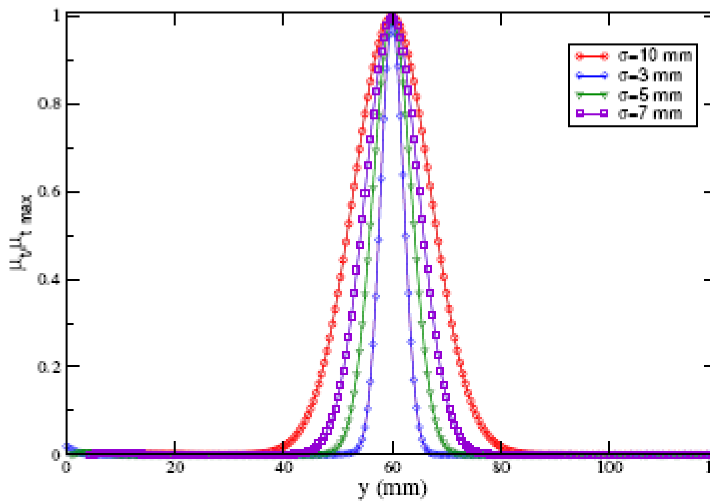


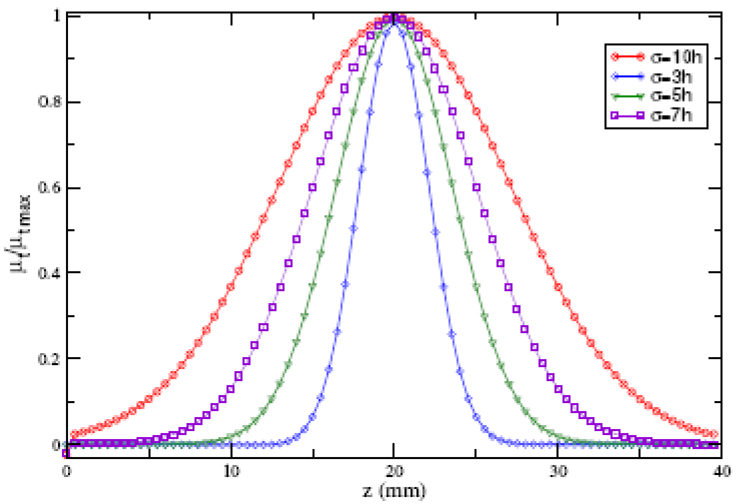
$$\beta = \frac{\pi}{4}$$

$$\frac{h}{L} = \frac{0.3}{40}$$

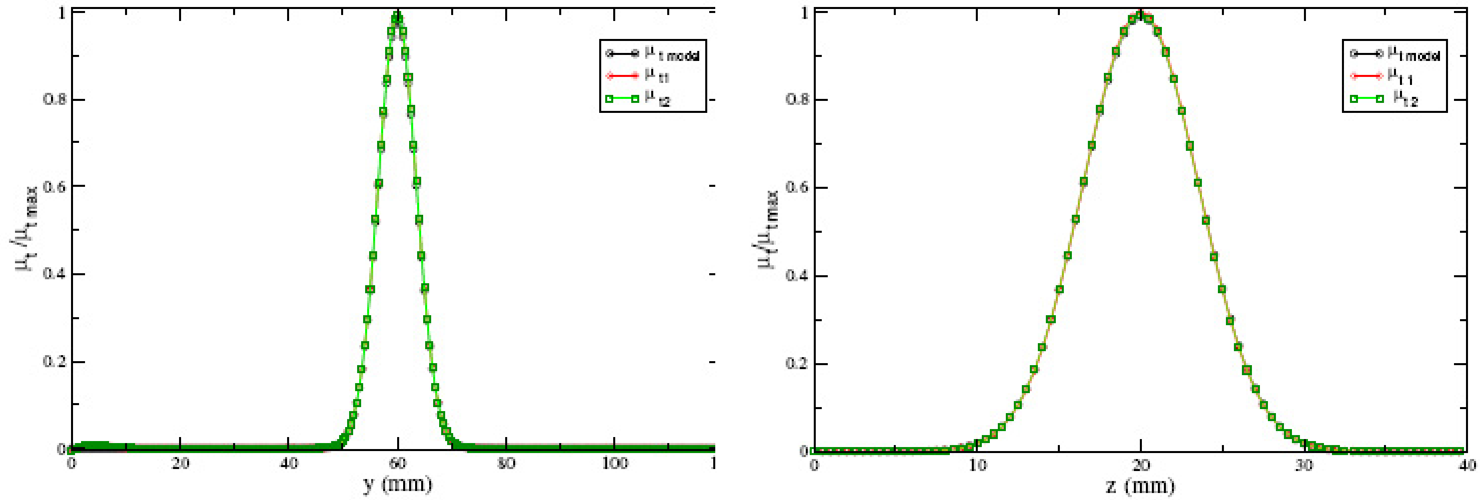
$$\frac{w}{L} = \frac{3}{40} \qquad \frac{w}{L} = \frac{5}{40} \qquad \frac{w}{L} = \frac{7}{40} \qquad \frac{w}{L} = \frac{10}{40}$$

Profiles of the reconstruction of a Gaussian

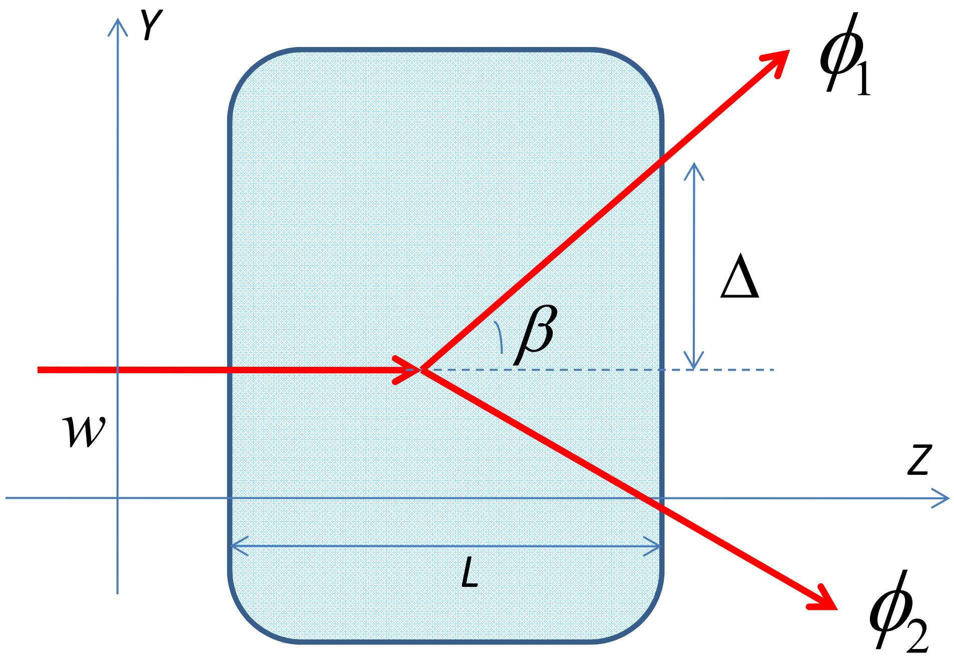




Real-space formula (for Gaussians)



Simultaneous reconstruction of absorption and scattering



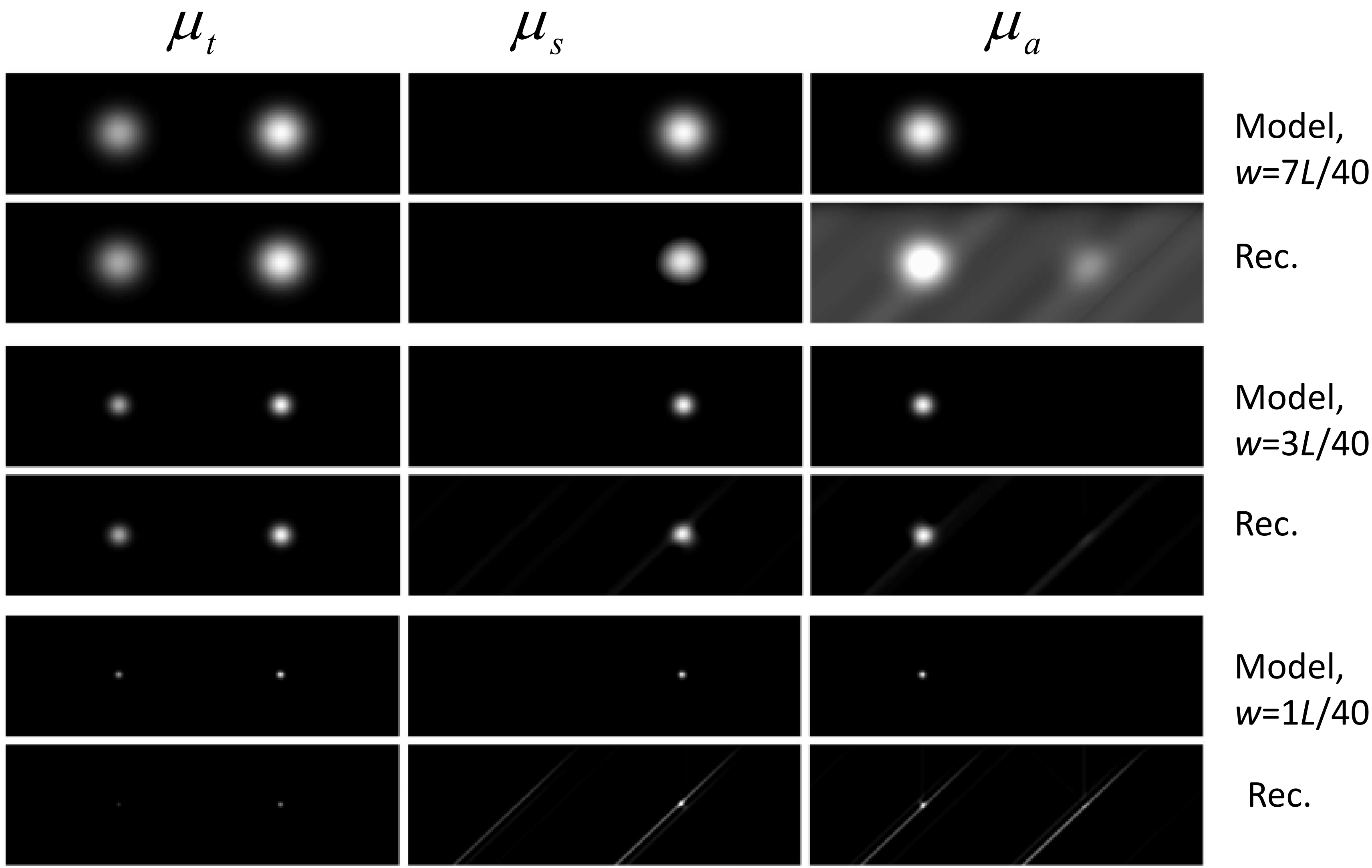
(b) $\mu_s \neq \text{const}$ (and is unknown)

$$\int_{\text{SSR}} \mu_t[\mathbf{r}(\ell)]d\ell - \ln\left[\frac{\mu_s(\mathbf{R_{12}})}{\langle \mu_s \rangle}\right] = \phi(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}_2, \hat{\mathbf{s}}_2)$$

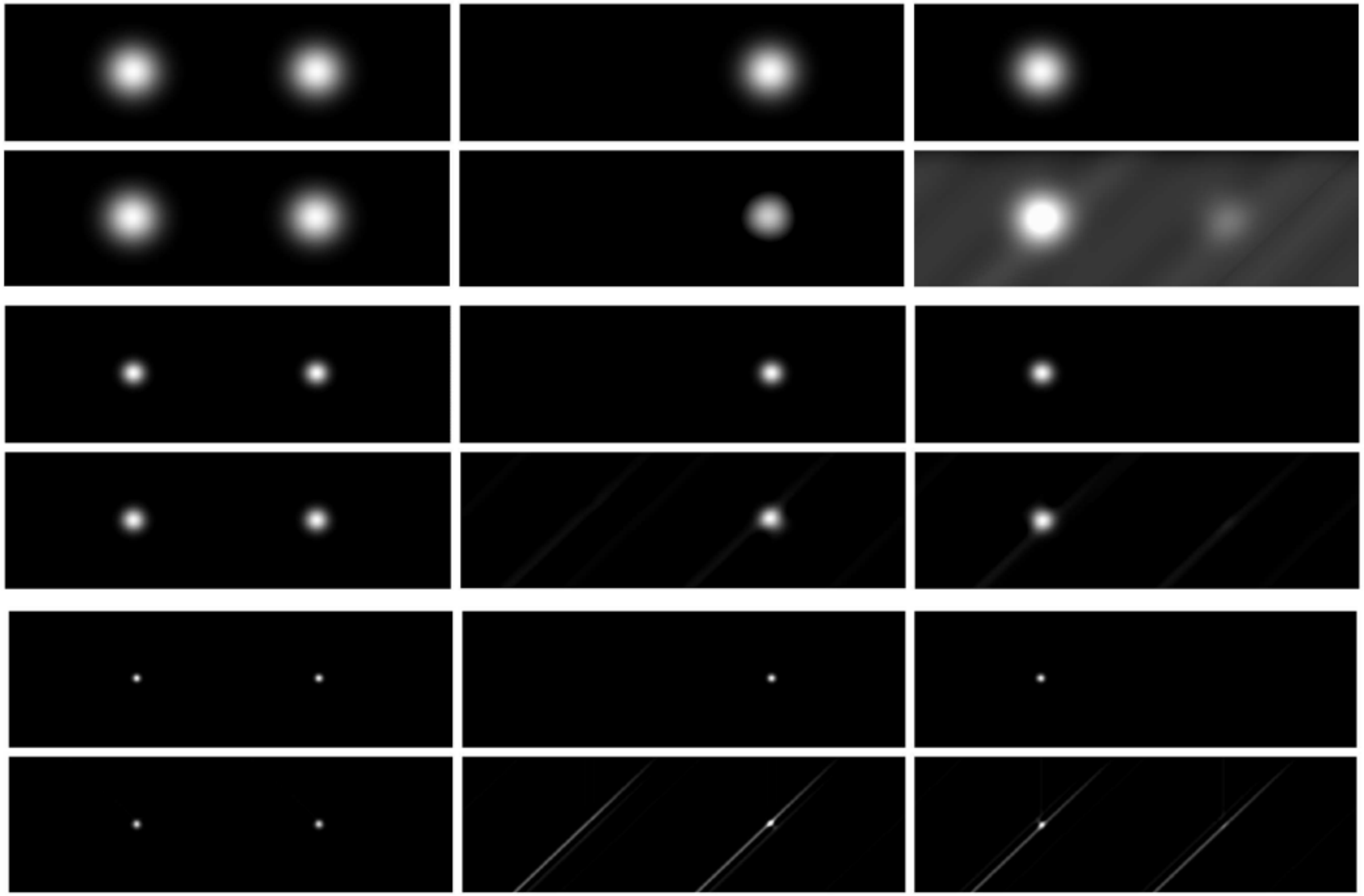
$$\phi = \phi_1 - \phi_2$$

Fourier-space image reconstruction formula for two rays

$$\tilde{\mu}_t(k,z) = \frac{\sin\beta}{2}\left(ik - \frac{1}{ik}\frac{\partial^2}{\partial\Delta^2}\right)\tilde{\phi}(k,\Delta)\bigg|_{\Delta=(L-z)\,\mathrm{tg}\,\beta}$$



$$\frac{\bar{\mu}_s}{\bar{\mu}_a} = \frac{4}{3} \;;\; \delta\mu_{a,s} = \bar{\mu}_{a,s} \exp\left[-\frac{(\boldsymbol{r} - \boldsymbol{r}_{a,s})^2}{w^2} \right] \;;\; \frac{h}{L} = \frac{0.3}{40}$$

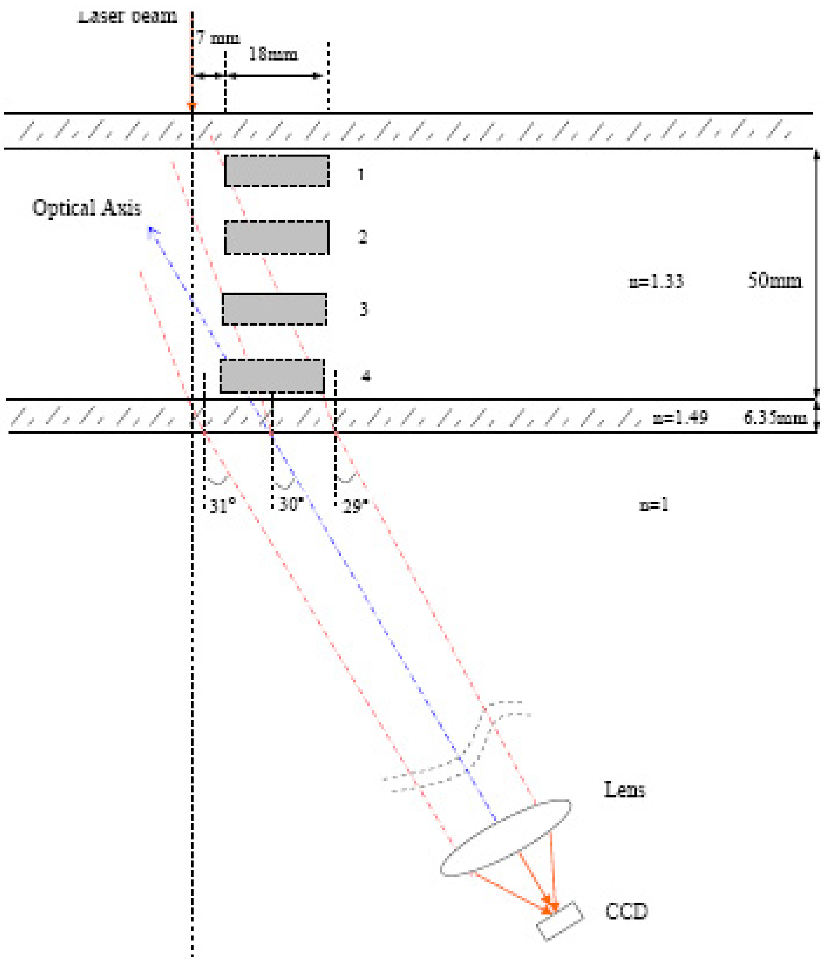


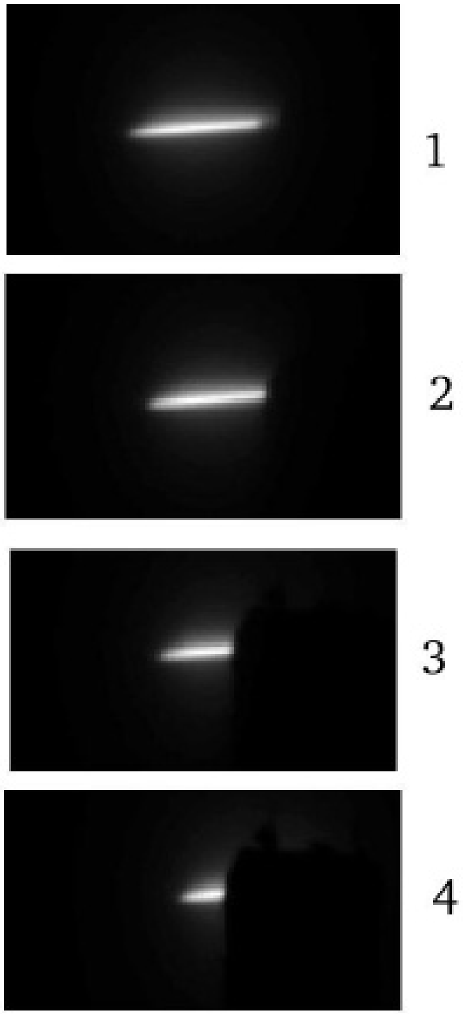
Same as above but for $\frac{\bar{\mu}_s}{\bar{\mu}_a} = 1$

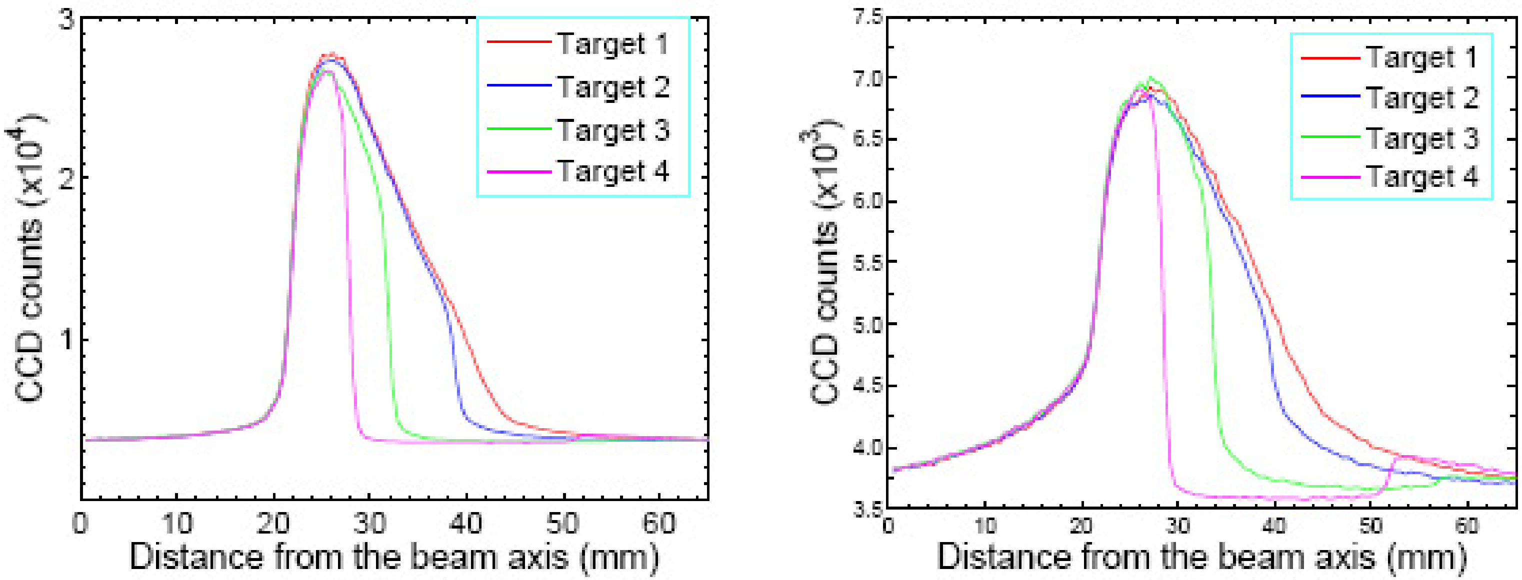
SUMMARY

- SSOT allows accurate quantitative reconstruction of the attenuation function.
- With additional measurements, scattering and absorption can be reconstructed separately
- Ill-posedness of the inverse problem is very mild.
- Tomographic imaging is feasible up to about six scattering lengths, with the noise-to-signal level of about 3% or less.

Preliminary Experiment







(a) (b)

Figure 6: Experimental measurements of the specific intensity as a function of the exit position on the slab surface for intralipid concentrations 0.02% (a) and 0.04% (b).