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Heating Rate and Impossibility of Negative Refraction

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In this talk, I will show that the general requirement for negative refraction in isotropic media, namely,

$$\mathrm{Im}(\epsilon\mu) < 0 \tag{1}$$

is in contradiction with the second law of thermodynamics. To this end, I compute the heating rate $q(\mathbf{r})$ in a magnetically and electrically polarizable medium. The computation is based on the fundamental expression

$$q = \langle \mathbf{J} \cdot \mathbf{E} \rangle \ , \tag{2}$$

where $\mathbf{E}$ is the electric field and

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M} \tag{3}$$

is the total current induced in the medium (we assume that there are no *external* currents or charges). The expression (2) is valid in any stationary state, in particular, for monochromatic fields, and the brackets $\langle \ldots \rangle$ denote time averaging.

The result obtained by applying (2) is different from the standard expression. I claim that the explanation for this discrepancy is that the Poynting vector in a magnetically polarizable medium must be defined by

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \tag{4}$$

rather than by the commonly used formula

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \tag{5}$$

To see that this is indeed, the case, we first evaluate the expression (2) for monochromatic fields of frequency ω in a linear, isotropic, homogeneous medium characterized by scalar complex permeability and permittivity $\epsilon(\omega)$ and $\mu(\omega)$. The derivation will be valid everywhere except on the medium boundary. The monochromatic fields have the general form

$$\mathbf{E} = \mathrm{Re}\left[\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}\right], \ \mathbf{D} = \mathrm{Re}\left[\mathbf{D}_\omega(\mathbf{r})e^{-i\omega t}\right], \ \mathbf{H} = \mathrm{Re}\left[\mathbf{H}_\omega(\mathbf{r})e^{-i\omega t}\right], \ \mathbf{B} = \mathrm{Re}\left[\mathbf{B}_\omega(\mathbf{r})e^{-i\omega t}\right], \tag{6}$$

where $\mathbf{D}_\omega = \epsilon(\omega)\mathbf{E}_\omega$, $\mathbf{B}_\omega = \mu(\omega)\mathbf{H}_\omega$, $c\nabla \times \mathbf{E}_\omega = i\omega\mathbf{B}_\omega$ and $c\nabla \times \mathbf{H}_\omega = -i\omega\mathbf{D}_\omega$. The current (except at the medium surface) can also be written in a similar form, namely,

$$\mathbf{J} = \mathrm{Re}\left[\mathbf{J}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \tag{7}$$

where

$$\mathbf{J}_\omega = \frac{1}{4\pi}\left(c\nabla \times \mathbf{B}_\omega + i\omega\mathbf{E}_\omega\right) \ . \tag{8}$$

We then write

$$q^{(V)} = \langle \mathbf{J}\cdot\mathbf{E}\rangle = \frac{1}{2}\mathrm{Re}(\mathbf{J}_\omega\cdot\mathbf{E}^*_\omega) = \frac{1}{8\pi}\mathrm{Re}\left[i\omega\mathbf{E}_\omega\cdot\mathbf{E}^*_\omega + c(\nabla\times\mathbf{B}_\omega)\cdot\mathbf{E}^*_\omega\right]$$
$$= \frac{c}{8\pi}\mathrm{Re}\left[\mu(\omega)(\nabla\times\mathbf{H}_\omega)\cdot\mathbf{E}^*_\omega\right] = \frac{c}{8\pi}\mathrm{Re}\left[\frac{-i\omega}{c}\mu(\omega)\epsilon(\omega)\mathbf{E}_\omega\cdot\mathbf{E}^*_\omega\right] = \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \quad (9)$$

Let us now adopt the definition (4) for the Poynting vector and compute $q^{(V)}$ from $q^{(V)} = -\langle\nabla\cdot\mathbf{S}\rangle$. We have:

$$q^{(V)} = -\nabla\cdot\langle\mathbf{S}\rangle = \frac{c}{8\pi}\mathrm{Re}\left[-\nabla\cdot(\mathbf{E}^*_\omega\times\mathbf{B}_\omega)\right] = \frac{c}{8\pi}\mathrm{Re}\left[\mathbf{E}^*_\omega\cdot(\nabla\times\mathbf{B}_\omega) - \mathbf{B}_\omega\cdot(\nabla\times\mathbf{E}^*_\omega)\right]$$
$$= \frac{c}{8\pi}\mathrm{Re}\left[\mathbf{E}^*_\omega\cdot(\nabla\times\mu(\omega)\mathbf{H}_\omega) + \frac{i\omega}{c}\mathbf{B}_\omega\cdot\mathbf{B}^*_\omega\right] = \frac{c}{8\pi}\mathrm{Re}\left[\frac{-i\omega}{c}\mu(\omega)\mathbf{E}^*_\omega\cdot\mathbf{D}_\omega\right] = \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \quad (10)$$

We thus have derived the same formula for $q^{(V)}$, namely,

$$q^{(V)} = \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \quad (11)$$

by both methods. We can conclude that if $\mathbf{S}$ is defined by (4), the two definition of the heating rate, $q^{(V)} = -\langle\nabla\cdot\mathbf{S}\rangle$ and $q^{(V)} = \langle\mathbf{J}\cdot\mathbf{E}\rangle$ are equivalent and we have the statement of local energy conservation which, in the stationary case, reads

$$\langle\mathbf{J}\cdot\mathbf{E}\rangle + \langle\nabla\cdot\mathbf{S}\rangle = 0 \ . \quad (12)$$

The derivation of the heating rate was so far restricted to points inside the medium. The additional surface term $q^{(S)}$ can be computed in a similar fashion. Both derivations yield the result

$$q^{(S)} = \frac{c}{8\pi}\mathrm{Re}\left[(1-\mu)(\hat{\mathbf{n}}\times\mathbf{H}_\omega)\cdot\mathbf{E}^*_\omega\right] \ . \quad (13)$$

The total heat absorbed by the body per unit time, Q, is given by

$$Q = \int_V q^{(V)}(\mathbf{r})d^3r + \oint_S q^{(S)}(\mathbf{r})d^2r \ , \quad (14)$$

where the first integral is evaluated over the body volume and the second over the surface. It is easy to see that this quantity is the same as in the conventional theory.

We now turn to the thermodynamic considerations. It is instructive to introduce the "accessible heat". This is the heat (either positive or negative) which can be transferred from the body to a heat reservoir on a time scale which is short compared to the time scales associated with heat diffusion in the body. Obviously, this is the heat generated at the surface.

Let $Q^{(S)} \equiv \oint_S q^{(S)}(\mathbf{r})d^2r = Q^{(S)}_+ - Q^{(S)}_-$. Here $Q^{(S)}_+$ is obtained by integration over the surface areas where $q^{(S)}(\mathbf{r})$ is positive and $Q^{(S)}_-$ is obtained by integration over the surface areas where $q^{(S)}(\mathbf{r})$ is negative. Let us further assume that the material exhibits negative refraction and $q^{(V)}(\mathbf{r})$ is negative, so the heat generated in the volume, $Q^{(V)} \equiv \int_V q^{(V)}(\mathbf{r})d^3r$ is also negative. Then we have $Q = -|Q^{(V)}| + Q^{(S)}_+ - Q^{(S)}_-$ or, equivalently, $Q^{(S)}_+ = Q + |Q^{(V)}| + Q^{(S)}_- > Q$. Thus, the positive accessible heat is greater than the total heat absorbed in

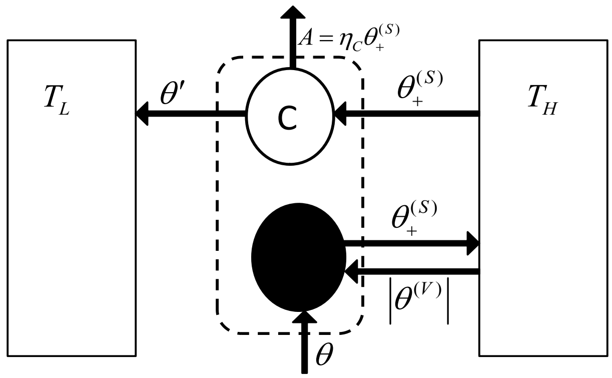

Fig. 1. A cyclic process involving a negative refractive index material that violates the Carnot theorem. The black oval represents a negative refraction medium and the white oval is an ideal Carnot engine.

the body. I will now demonstrate that this contradicts the Carnot theorem and, moreover, can be used to create a *perpetuum mobile* of the second kind. To see that this is, indeed the case, consider the cyclic process shown in the Fig. 1.

In this cycle, the following events happen: (1) A negative-refraction sample represented by the black oval (referred to as the "body" below) at the initial temperature T_H is irradiated for a period of time Δt which is short compared to the time scales associated with heat diffusion in the body, yet long compared to the electromagnetic oscillations period, so that radiation is almost monochromatic. The body absorbs the energy $\theta = Q\Delta t$ from the radiation field. (2) The body is brought in contact with a heat reservoir at the temperature T_H which has very high heat conductivity; the amount of thermal energy $\theta_+^{(S)} = Q_+^{(S)}\Delta t$ generated at the body's surface is transferred adiabatically to this reservoir. (3) The body is disconnected from the reservoir and heat diffusion takes place in the body until the equilibrium temperature $T' < T_H$ is reached. (4) An ideal Carnot engine is operated for one cycle between the hot reservoir and a colder reservoir whose temperature is $T_L < T_H$. The Carnot engine absorbs the heat $\theta_+^{(S)}$ from the hot reservoir, makes useful work $A = \eta_C \theta_+^{(S)}$ and rejects some amount of heat $\theta' = \theta_+^{(S)} - A$ to the cold reservoir. Here $\eta_C = 1 - T_L/T_H$ is the efficiency of an ideal Carnot engine. (5) The body is again brought in contact with the hot reservoir; now the heat $|\theta^{(V)}|$ flows back from the hot reservoir to the body. In the end of this process, the body has the temperature T_H. (6) We disconnect the body from the hot reservoir. Now the cycle is complete and the system has returned to its original state.

The net effect of the above thermodynamic transformation is the following: The electromagnetic field has done the work θ on the body which was immediately dissipated into heat θ; we then converted this heat into the useful work A. The overall efficiency of this process is

$$\eta = \frac{A}{\theta} = \eta_C \frac{\theta_+^{(S)}}{\theta_+^{(S)} - |\theta^{(V)}|} = \eta_C \frac{1}{1 - |\theta^{(V)}|/\theta_+^{(S)}} > \eta_C \tag{15}$$

in violation of the Carnot theorem. Moreover, we can operate a *perpetuum mobile* of the second kind if $A > \theta$ or, equivalently, $\eta > 1$. This is achieved if $T_L/T_H < |\theta^{(V)}|/\theta_+^{(S)}$. There is no physical reason why this condition can not be met. Therefore, I conclude that negative refraction is impossible.