Radiative transport for two-photon light

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We consider the propagation of two-photon light in a random medium. We show that the Wigner transform of the two-photon amplitude obeys an equation that is analogous to the radiative transport equation for classical light. Using this result, we investigate the propagation of an entangled photon pair.

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I. INTRODUCTION

The propagation of light in disordered media, including clouds, colloidal suspensions, and biological tissues, is generally considered within the framework of classical optics [1]. However, recent experiments have demonstrated the existence of novel effects in multiple light scattering, in which the quantized nature of the electromagnetic field is manifest. These include (i) the transport of quantum noise through random media [2], (ii) the observation of spatial correlations in multiply scattered squeezed light [3,4], (iii) the measurement of two-photon speckle patterns and the observation of nonexponential statistics for two-photon correlations [5,6], and (iv) the finding that interference survives averaging over disorder, as evidenced by photon correlations exhibiting both antibunching and anyonic symmetry [7,8]. Thus there is an interplay between quantum interference and interference due to multiple scattering that is of fundamental interest [9–15] and considerable applied importance. Indeed, applications to spectroscopy [16], two-photon imaging [17–26], and quantum communication [27–29] have been reported.

In the multiple-scattering regime, the radiative transport equation (RTE) governs the propagation of light in random media [1]. The RTE is a conservation law that accounts for gains and losses of electromagnetic energy due to scattering and absorption. The physical quantity of interest is the specific intensity $I(\mathbf{r},\hat{\mathbf{k}})$, defined as the intensity at the position $\mathbf{r}$ in the direction $\hat{\mathbf{k}}$. The specific intensity obeys the RTE

$$\begin{aligned}\hat{\mathbf{k}}\cdot\nabla_{\mathbf{r}}I(\mathbf{r},\hat{\mathbf{k}})&+(\mu_a+\mu_s)I(\mathbf{r},\hat{\mathbf{k}})\\ &=\mu_s\int d^2k'[p(\hat{\mathbf{k}}',\hat{\mathbf{k}})I(\mathbf{r},\hat{\mathbf{k}}')-p(\hat{\mathbf{k}},\hat{\mathbf{k}}')I(\mathbf{r},\hat{\mathbf{k}})],\end{aligned}\tag{1}$$

which we have written in its stationary form. Here μ_a and μ_s are the absorption and scattering coefficients of the medium and p is the phase function. We note that although the RTE is often viewed as phenomenological, it is derivable from the scattering theory of electromagnetic waves in a random medium [1,30,31].

The propagation of two-photon light is generally considered either in free space or, in some cases, with account of diffraction [32,33]. However, understanding the interaction of light with matter is central to applications in both imaging and quantum information. In this paper, we consider the propagation of two-photon light in a random medium. We show that the averaged Wigner transform of the two-photon amplitude obeys an equation that is analogous to the RTE for classical light. Using this result, we investigate the propagation of an entangled photon pair in a random medium. In this sense, our work builds on the well-known duality between partially coherent and partially entangled light [33].

The remainder of this paper is organized as follows. In Sec. II we recall some important facts about the propagation of two-photon light. We also introduce the Wigner transform of the two-photon amplitude and derive the Liouville equation obeyed by the Wigner transform. In Sec. III we use this result to obtain the RTE for two-photon light, which is then applied to study the propagation of an entangled pair in Sec. IV. Our conclusions are formulated in Sec. V.

II. TWO-PHOTON LIGHT

Let $|\psi\rangle$ denote a two-photon state. We define the second-order coherence function as the normally ordered expectation of field operators:

$$\begin{aligned}\Gamma^{(2)}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\langle\psi|&\widehat{E}^-(\mathbf{r}_1,t_2)\widehat{E}^-(\mathbf{r}_2,t_2)\\ &\times\widehat{E}^+(\mathbf{r}_2,t_2)\widehat{E}^+(\mathbf{r}_1,t_1)|\psi\rangle,\end{aligned}\tag{2}$$

where $\widehat{E}^-$ and $\widehat{E}^+$ are the negative- and positive-frequency components of the electric-field operator with $\widehat{E}^-=[\widehat{E}^+]^\dagger$. In a material medium with dielectric permittivity ε, the field operator $\widehat{E}^+$ obeys the wave equation [34,35]

$$\nabla^2\widehat{E}^+-\frac{\varepsilon(\mathbf{r})}{c^2}\frac{\partial^2\widehat{E}^+}{\partial t^2}=0\,.\tag{3}$$

Here the medium is taken to be nonabsorbing, so that ε is purely real, positive, and frequency independent.

The quantity $\Gamma^{(2)}$ is proportional to the probability of detecting one photon at $\mathbf{r}_1$ at time t_1 and a second photon at $\mathbf{r}_2$ at time t_2. It can be measured in a Hanbury Brown–Twiss interferometer [36]. For the two-photon state $|\psi\rangle$, it can be

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seen that $\Gamma^{(2)}$ factorizes [37] as follows:

$$\begin{aligned}\Gamma^{(2)}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) &= \sum_n \langle\psi|\widehat{E}^-(\mathbf{r}_1,t_1)\widehat{E}^-(\mathbf{r}_2,t_2)|n\rangle \\ &\quad\times \langle n|\widehat{E}^+(\mathbf{r}_2,t_2)\widehat{E}^+(\mathbf{r}_1,t_1)|\psi\rangle \\ &= \langle\psi|\,\widehat{E}^-(\mathbf{r}_1,t_1)\widehat{E}^-(\mathbf{r}_2,t_2)\,|0\rangle \\ &\quad\times \langle 0|\,\widehat{E}^+(\mathbf{r}_2,t_2)\widehat{E}^+(\mathbf{r}_1,t_1)\,|\psi\rangle \\ &= |\Phi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)|^2 .\end{aligned} \tag{4}$$

Here $\{|n\rangle\}$ denotes a complete set of Fock states, $|0\rangle$ is the vacuum state, and the two-photon amplitude Φ is defined by

$$\Phi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = \langle 0|\,\widehat{E}^+(\mathbf{r}_1,t_1)\widehat{E}^+(\mathbf{r}_2,t_2)\,|\psi\rangle . \tag{5}$$

Evidently, Φ satisfies the pair of wave equations

$$\nabla^2_{\mathbf{r}_j}\Phi - \frac{\varepsilon(\mathbf{r}_j)}{c^2}\frac{\partial^2\Phi}{\partial t_j^2} = 0\,, \quad j = 1,2\,, \tag{6}$$

which follows from the fact that $\widehat{E}^+$ obeys the wave equation (3). We note that (6) is the analog of the Wolf equations for two-photon light [38]. We will find it convenient to introduce the Fourier transform of the amplitude Φ, which is given by

$$\widetilde{\Phi}(\mathbf{r}_1,\omega_1;\mathbf{r}_2,\omega_2) = \int dt_1 dt_2 e^{i(\omega_1 t_1+\omega_2 t_2)}\Phi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\,. \tag{7}$$

Equation (6) then becomes

$$\nabla^2_{\mathbf{r}_j}\widetilde{\Phi} + k_j^2\varepsilon(\mathbf{r}_j)\widetilde{\Phi} = 0, \quad j = 1,2\,, \tag{8}$$

where $k_j = \omega_j/c$. It is important to note that if $\widetilde{\Phi}$ factorizes into a product of two functions which depend upon $\mathbf{r}_1$ and $\mathbf{r}_2$ separately, then the two-photon state $|\psi\rangle$ is not entangled. In contrast, a *fully entangled state* is not separable and corresponds to $\widetilde{\Phi}(\mathbf{r}_1,\mathbf{r}_2) \propto \delta(\mathbf{r}_1 - \mathbf{r}_2)$.

We now consider the Wigner transform of $\widetilde{\Phi}$, which is defined by

$$W(\mathbf{r},\mathbf{k}) = \int d^3r' e^{i\mathbf{k}\cdot\mathbf{r}'}\widetilde{\Phi}(\mathbf{r}-\mathbf{r}'/2,\omega_1;\mathbf{r}+\mathbf{r}'/2,\omega_2)\,. \tag{9}$$

We will see that the Wigner transform plays a role that is analogous to the specific intensity in radiative transport theory. To derive the equation obeyed by W, we subtract the pair of equations (8) and change variables according to

$$\mathbf{r}_1 = \mathbf{r} - \mathbf{r}'/2\,, \quad \mathbf{r}_2 = \mathbf{r} + \mathbf{r}'/2\,. \tag{10}$$

We find that

$$\begin{aligned}\mathbf{k}\cdot\nabla_{\mathbf{r}}W + \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{r}}\widetilde{\varepsilon}(\mathbf{p})\big[k_1^2W(\mathbf{r},\mathbf{k}+\mathbf{p}/2) \\ - k_2^2W(\mathbf{r},\mathbf{k}-\mathbf{p}/2)\big] = 0\,,\end{aligned} \tag{11}$$

where

$$\widetilde{\varepsilon}(\mathbf{p}) = \int d^3r e^{i\mathbf{p}\cdot\mathbf{r}}\varepsilon(\mathbf{r}) \tag{12}$$

is the Fourier transform of ε. We note that (11) is an exact result.

III. TWO-PHOTON RTE

We now proceed to derive the RTE for two-photon light. To this end, we consider a statistically homogeneous random medium and assume that the susceptibility η is a Gaussian random field with correlations

$$\langle\eta(\mathbf{r})\rangle = 0\,, \tag{13}$$

$$\langle\eta(\mathbf{r})\eta(\mathbf{r}')\rangle = C(|\mathbf{r}-\mathbf{r}'|)\,. \tag{14}$$

Here η is related to the dielectric permittivity by $\varepsilon = 1 + 4\pi\eta$, C is the two-point correlation function, and $\langle\cdots\rangle$ denotes statistical averaging. Let L denote the distance over which the field propagates and ξ the correlation length over which C decays at large distances. We introduce a small parameter $\epsilon = 1/(k_0L) \ll 1$, where k_0 is the spatial bandwidth, and suppose that the fluctuations in η are sufficiently weak that $C,\xi/L = O(\epsilon)$. We then rescale the spatial variables according to $\mathbf{r}_1 \to \mathbf{r}_1/\epsilon$, $\mathbf{r}_2 \to \mathbf{r}_2/\epsilon$ and define the scaled two-photon probability amplitude $\Phi_\epsilon(\mathbf{r}_1,\omega_1;\mathbf{r}_2,\omega_2) = \widetilde{\Phi}(\mathbf{r}_1/\epsilon,\omega_1;\mathbf{r}_2/\epsilon,\omega_2)$, so that (8) becomes

$$\epsilon^2\nabla^2_{\mathbf{r}_j}\Phi_\epsilon + k_j^2\Phi_\epsilon = -4\pi k_j^2\sqrt{\epsilon}\eta(\mathbf{r}_j/\epsilon)\Phi_\epsilon, \quad j = 1,2\,, \tag{15}$$

where we have introduced a rescaling of η to be consistent with the assumption that the fluctuations are of size $O(\epsilon)$. If we denote by W_ϵ the Wigner transform of Φ_ϵ, defined according to (9), then (11) becomes

$$\mathbf{k}\cdot\nabla_{\mathbf{r}}W_\epsilon + \frac{i}{2\epsilon}\big(k_1^2 - k_2^2\big)W_\epsilon + \frac{1}{\sqrt{\epsilon}}\mathscr{L}W_\epsilon = 0\,, \tag{16}$$

where

$$\begin{aligned}\mathscr{L}W_\epsilon = 2\pi i\int\frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{r}/\epsilon}\widetilde{\eta}(\mathbf{p})\big[k_1^2W_\epsilon(\mathbf{r},\mathbf{k}+\mathbf{p}/2) \\ - k_2^2W_\epsilon(\mathbf{r},\mathbf{k}-\mathbf{p}/2)\big]\,.\end{aligned} \tag{17}$$

We now consider the asymptotics of the Wigner transform in the homogenization limit $\epsilon \to 0$. This corresponds to the regime of weak fluctuations. Following standard procedures [30], we introduce a two-scale expansion for W_ϵ of the form

$$\begin{aligned}&W_\epsilon(\mathbf{r},\mathbf{R},\mathbf{k}) \\ &= W_0(\mathbf{r},\mathbf{R},\mathbf{k}) + \sqrt{\epsilon}W_1(\mathbf{r},\mathbf{R},\mathbf{k}) + \epsilon W_2(\mathbf{r},\mathbf{R},\mathbf{k}) + \cdots\,,\end{aligned} \tag{18}$$

where $\mathbf{R} = \mathbf{r}/\epsilon$ is a fast variable. Next we suppose that $\gamma = (k_1^2 - k_2^2)/(2k\epsilon) = O(1)$, which corresponds to working in the high-frequency regime. By averaging over the fluctuations on the fast scale, it can be seen that $\langle W_0\rangle$, which we denote by $\mathcal{I}$, obeys the equation

$$\hat{\mathbf{k}}\cdot\nabla_{\mathbf{r}}\mathcal{I}(\mathbf{r},\hat{\mathbf{k}}) + (\sigma_a + \sigma_s)\mathcal{I}(\mathbf{r},\hat{\mathbf{k}}) = \sigma_s\int d^2k' f(\hat{\mathbf{k}},\hat{\mathbf{k}}')\mathcal{I}(\mathbf{r},\hat{\mathbf{k}}')\,. \tag{19}$$

Here the coefficients σ_a, σ_s and the scattering kernel f are defined by

$$\sigma_a = i\gamma, \tag{20a}$$

$$\sigma_s = k_1^2k_2^2\int\widetilde{C}(k(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))d^2k', \tag{20b}$$

$$f(\hat{\mathbf{k}},\hat{\mathbf{k}}') = \frac{\widetilde{C}(k(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}{\int d^2k'\widetilde{C}(k(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}\,, \tag{20c}$$

where $f(\hat{\mathbf{k}},\hat{\mathbf{k}}')$ is normalized so that $\int f(\hat{\mathbf{k}},\hat{\mathbf{k}}')d^2k' = 1$ for all $\hat{\mathbf{k}}$. We note that this normalization is consistent with the statistical homogeneity of the random medium, since $\widetilde{C}(\mathbf{k}-\mathbf{k}')$ depends only upon the quantity $\mathbf{k}\cdot\mathbf{k}'$. We will refer to (19) as the two-photon RTE and the quantity $\mathcal{I}$ as the two-photon specific intensity.

Some remarks on (19) are now called for. Evidently, (19) bears some resemblance to the classical RTE (1). However, it differs from the latter both mathematically and in its physical interpretation. In particular, the quantity $\mathcal{I}$, in contrast to the specific intensity, is not real valued and is not directly measurable. Nevertheless, by inversion of the Fourier transform (9), we find that $\mathcal{I}$ is related to the two-photon amplitude by means of the formula

$$\langle\widetilde{\Phi}(\mathbf{r}_1,\mathbf{r}_2)\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\mathcal{I}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\hat{\mathbf{k}}\right), \tag{21}$$

where the dependence on the frequencies ω_1 and ω_2 has not been indicated. We note that Φ, in turn, is related to a physically measurable quantity, namely, the counting rate in a Hanbury Brown–Twiss experiment [36]. In addition, the coefficient σ_a is complex valued and does not lead to absorption of energy, as is the case for the absorption coefficient μ_a in the RTE. Indeed, as previously indicated, we assume that the medium is nonabsorbing. This assumption, along with the approximations of weak scattering, statistical homogeneity, and weak disorder are standard in the theory of radiative transport [1].

IV. PROPAGATION OF AN ENTANGLED TWO-PHOTON STATE

We now explore some physical consequences of the two-photon RTE. In particular, we examine the propagation of an entangled photon pair. We begin with the case of a *deterministic* medium in which the permittivity ε is constant. We consider the half-space $z \geqslant 0$ and assume that the two-photon Wigner transform $\mathcal{I}_0$ is specified on the disk of radius a in the plane $z=0$ in the direction $\hat{\mathbf{k}}\cdot\hat{\mathbf{z}} > 0$, as illustrated in Fig. 1. That is,

$$\mathcal{I}_0(\mathbf{r},\hat{\mathbf{k}}) = \begin{cases} A\delta(k-k_0) & \text{if} \quad \hat{\mathbf{k}}\cdot\hat{\mathbf{z}} > 0 \quad \text{and} \quad |\boldsymbol{\rho}| \leqslant a\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{22}$$

where A is a constant and $\boldsymbol{\rho}$ is the transverse coordinate in the $z=0$ plane. Making use of (21), it is readily seen that

$$\widetilde{\Phi}(\boldsymbol{\rho}_1,0;\boldsymbol{\rho}_2,0) = \begin{cases} 2\pi k_0^2 A\frac{\sin(k_0|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)}{k_0|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|} & \text{if} \quad |\boldsymbol{\rho}_{1,2}| \leqslant a\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{23}$$

which corresponds to a transversely entangled two-photon state. To propagate $\mathcal{I}$ into the $z>0$ half-space, we make use of the formula

$$\mathcal{I}(\mathbf{r},\hat{\mathbf{k}}) = \int d^2k' \int_{z'=0} d^2r'\, \hat{\mathbf{z}}\cdot\hat{\mathbf{k}}' G(\mathbf{r},\hat{\mathbf{k}};\mathbf{r}',\hat{\mathbf{k}}')\mathcal{I}_0(\mathbf{r}',\hat{\mathbf{k}}'). \tag{24}$$

Here G is the Green's function for the two-photon RTE (11), which is given by

$$G(\mathbf{r},\hat{\mathbf{k}};\mathbf{r}',\hat{\mathbf{k}}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\delta(\hat{\mathbf{k}}-\hat{\mathbf{k}}')\delta\left(\hat{\mathbf{k}}-\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right). \tag{25}$$

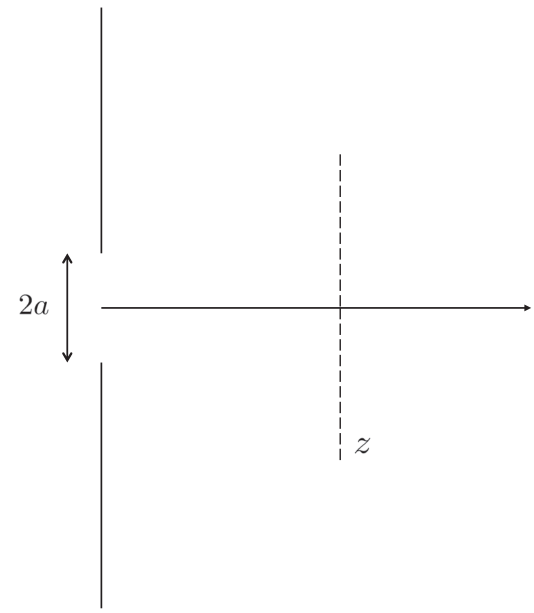

FIG. 1. Illustrating the geometry. A circular aperture of radius a is located in the $z=0$ plane.

We can now compute the two-photon probability amplitude $\widetilde{\Phi}$. For simplicity, we assume that $k_1=k_2=k_0$ and that the points of observation $\mathbf{r}_1$ and $\mathbf{r}_2$ are on axis, with $\mathbf{r}_1=\mathbf{r}_2=(0,z)$. Carrying out the integrations in (24) and making use of (21), we find that

$$\widetilde{\Phi}(0,z;0,z) = A\left(\frac{k_0}{2\pi}\right)^2 \tan^{-1}\left(\frac{a}{2z}\right). \tag{26}$$

In Fig. 2 we plot the z dependence of $\widetilde{\Phi}$, which illustrates the propagation of an entangled photon pair. We note that, in principle, the the above result can be obtained directly from the wave equation, thus bypassing the RTE. This is not surprising, in view of the results of [32,33] which are also obtained under conditions of free-space propagation. Finally, we observe that the diagonal part of the coherence function $\Gamma^{(2)}(\mathbf{r},\mathbf{r}) = |\widetilde{\Phi}(\mathbf{r},\mathbf{r})|^2$ is proportional to the probability of

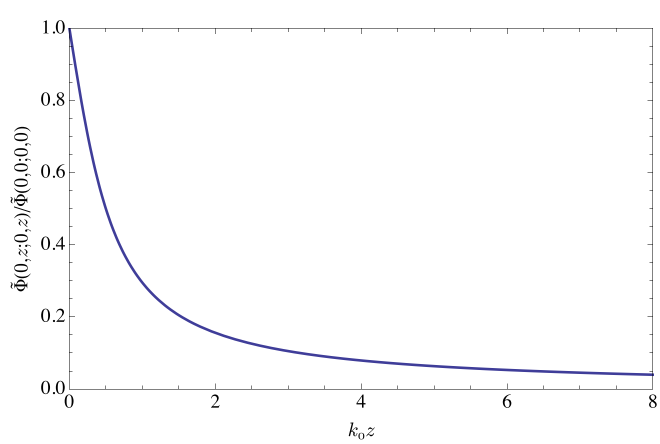

FIG. 2. (Color online) Dependence of the two-photon amplitude $\widetilde{\Phi}$ on the distance of propagation z in free space, where $k_0 a = 1$, with a the radius of the aperture.

two-photon absorption at the point $\mathbf{r}$, which is a physically observable quantity.

Next we consider the case of a *random* medium. In this situation, the full machinery of radiative transport is required, which allows for the description of the intertwined effects of quantum interference and interference due to multiple scattering. For simplicity, we make use of the diffusion approximation (DA) to the RTE, which is widely utilized in applications. The DA neglects the angular dependence of the Green's function for the RTE [39]. It holds in the limit of strong scattering and at large distances from the source [1]. Within the accuracy of the DA, the Green's function for the RTE is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi D|\mathbf{r}-\mathbf{r}'|}, \tag{27}$$

where the diffuse wave number $\kappa = \sqrt{3\sigma_a \ell^*}$ and the diffusion constant $D = 1/3c\ell^*$. The transport length ℓ^* and the scattering anisotropy g are defined by

$$\ell^* = 1/[\sigma_a + (1-g)\sigma_s)], \quad g = \int \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' f(\hat{\mathbf{k}},\hat{\mathbf{k}}')d^2k'. \tag{28}$$

Carrying out the integrations in (21) and (24), we find that the average two-photon probability amplitude is given by

$$\begin{aligned}\langle\widetilde{\Phi}(\mathbf{r}_1,\mathbf{r}_2)\rangle &= \frac{aAk_0}{2D(2\pi)^2}\frac{\sin(k_0|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|} \\ &\quad\times \int_0^\infty \frac{dq}{\sqrt{q^2+\kappa^2}} J_1(qa) \\ &\quad\times J_0(q|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|/2)e^{-\sqrt{q^2+\kappa^2}(z_1+z_2)/2},\end{aligned} \tag{29}$$

where $\mathbf{r} = (\boldsymbol{\rho},z)$. In the on-axis configuration, we find that

$$\langle\widetilde{\Phi}(0,z;0,z)\rangle = \frac{Ak_0}{2D(2\pi)^2}[\sqrt{z^2+a^2} - z]. \tag{30}$$

To illustrate the propagation of the amplitude of an entangled photon pair, we show in Fig. 3 the z dependence of $\langle\widetilde{\Phi}(\mathbf{r},\mathbf{r})\rangle$, for various values of the off-axis distance ρ. We caution that the interpretation of this result requires some care. In particular, the decay of $\langle\widetilde{\Phi}(\mathbf{r},\mathbf{r})\rangle$ should not be interpreted as the "loss of entanglement" of the photon pair. Rather, the two-photon RTE can be considered as a conservation law for the two-photon specific intensity $\mathcal{I}$. On a related note, various measures of entanglement, including the Schmidt number and the entropy, can be constructed from the singular values σ_n of $\widetilde{\Phi}$, defined by $\widetilde{\Phi}^*\widetilde{\Phi}u_n = \sigma_n^2 u_n$, where u_n are the corresponding eigenfunctions [40,41]. Here the Schmidt number $K = 1/\sum_n \sigma_n^2$ and the entropy $S = -\sum_n \sigma_n \log\sigma_n$. We plan to explore the propagation of entanglement, as measured by K and S, in future work.

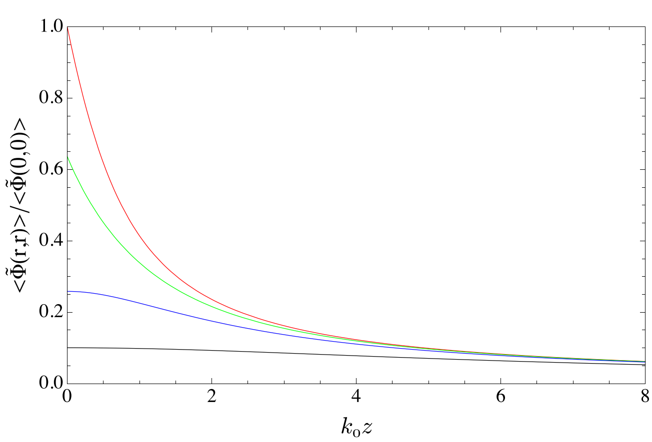

FIG. 3. (Color online) Dependence of the two-photon amplitude $\langle\widetilde{\Phi}\rangle$ on the distance of propagation z for different off-axis distances ρ, with $k_0\rho = 0,1,2,5$ (top to bottom). The diffuse wave number $\kappa = 0$ and $k_0 a = 1$, with a the radius of the aperture.

V. DISCUSSION

We close with a few remarks. It is possible to derive the analog of the RTE for single photons. Not surprisingly, this equation has the form of the classical RTE (1), a result whose derivation will be presented elsewhere.

The derivation of the two-photon RTE makes use of a two-scale asymptotic expansion. Alternatively, it may be possible to obtain this result from diagrammatic perturbation theory, as is the case for the classical RTE [1] and related transport equations for electronic systems [42,43]. This is a potentially interesting topic for future research.

The theory we have developed cannot be used to describe Anderson localization for two-photon light [44–47]. This is not unexpected, since for classical light, localization is not described by radiative transport theory [1].

Although in our model the electromagnetic field is quantized, the interaction of the field with the scattering medium is treated classically. It would be of interest to extend our results to the case in which the medium consists of a collection of two- or three-level atoms. In this manner, it should (in principle) be possible to understand the transfer of entanglement from the field to the medium [48]. Evidently, the calculations that we have presented do not account for this effect, since we have taken a macroscopic approach to the quantization of the field [34,35].

Finally, applications to imaging and communication theory may be envisioned. In the former case, there has been extensive use of the classical RTE for imaging in random media. It may be anticipated that experiments with two-photon light may enjoy some advantages, as has been suggested for the case of quantum optical coherence tomography [49,50]. In the latter case, there has been considerable interest in the use of quantum states of light for communication [27,29,51]. It would be of interest to understand the effect of a complex medium, such as the atmosphere, on the capacity of quantum information systems [28].

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