What is extinction? Operational definition of the extinguished power for plane waves and collimated beams

Vadim A. Markel[1]

Department of Radiology, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA

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ABSTRACT

We discuss an operational scheme for measuring the power extinguished by a single particle in terms of physical energy fluxes. Illumination by an infinite plane wave and a collimated Gaussian beam is considered. For the case of a collimated beam, consideration of extinguished power presents an apparent paradox, which is resolved in this paper. It is then shown that the extinguished power is measurable as a flow of energy for a narrow, collimated incident beam and a small scatterer. In this case, the extinguished power is simply removed from the transmitted beam. If we relax the above assumptions, definition of the extinguished power in terms of measurable energy fluxes is still possible but becomes more nuanced.

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1. Introduction

Extinction of waves by particles or potentials plays a fundamental role in the theory of scattering. For incident plane waves, one can introduce the absorption and scattering cross sections σ_{a} and σ_{s} and the extinction cross section $\sigma_{\mathrm{e}} = \sigma_{\mathrm{a}} + \sigma_{\mathrm{s}}$. Conventionally, σ_{e} is interpreted as a quantitative measure of how strongly an object *extinguishes*, that is, modifies and/or suppresses the incident wave. In particular, smallness of σ_{e} is a better indicator of invisibility of an object than smallness of σ_{s} [1]. It should be clarified however that the incident wave is not really modified in a scattering process; it is determined by the external sources of radiation and independent of the scatterer. Therefore, what exactly is quantified by the extinction cross section is a subtle question the answer to which may depend on the physical situation [2]. The goal of this paper is to clarify the meaning of extinction for some commonly encountered incident fields.

The extinguished power is surprisingly difficult to express in terms of physical energy fluxes since it involves the product of the incident and the scattered fields (the "cross term") whereas only the sum of these two fields enters the definition of any directly measurable flux. This has led historically to various paradoxes, including the classical extinction paradox whose complete understanding was achieved only recently [3, 4, 5] although an important insight was made by Brillouin in 1949 [6]. Another paradox related to extinction, which is much less known, is discussed and resolved below.

In addition to resolving the above-mentioned paradox, we will show that, in the case of incident collimated beams, the extinguished power can be defined in terms of a physical energy flux, which can be measured with one or at most two flat detectors. We say that such a definition is *operational*. The definition is in good agreement with the conventional interpretation of extinction: the extinguished power is literally removed from the beam. This result could, of course, be anticipated from the experimental point of view. However, its theoretical demonstration is subtle. We note that providing a similar direct measurement scheme for incident plane waves is also possible but somewhat problematic and involves several applicability conditions.

In the case of an incident collimated beam and not very forward-peaked scattering, the incident and the scattered waves overlap only in a relatively small region of space. As one could expect, an operational definition of extinction can be given more easily when the incident and the scattered waves overlap in some sense weakly. This does not mean that there is no interference between the two waves – extinction *is* interference. However, the condition of small overlap implies that a dominant part of the scattered power does not interfere with the incident wave. If this is so, the scattered power also becomes (approximately) a physical energy flux, although its measurement requires a 4π solid angle arrangement of detectors.

We start by briefly reviewing what is currently known about extinction. One basic result in this area is the optical theorem, which relates σ_{e} to the imaginary part of the scattering amplitude in the forward direction. However, a forward direction can be defined only for an incident plane wave. To circumvent this problem, generalizations of the optical theorem for various superpositions of incident plane waves or (equivalently) for incident structured beams have been derived [7, 8, 9, 10]. These generalizations are not as simple as the classical optical theorem and involve either Fourier-space or real-space integrals.

The generalization of the optical theorem that involves a real-space integral [7] (also see [11, 12] for the discretized case) has a more transparent physical interpretation. Namely, the extinguished power is expressed as the total work done by the incident field on the medium, i.e., on the induced electric current in the case of electromagnetic waves or on moving

vmarkel@upenn.edu (V.A. Markel)
ORCID(s): 0000-0002-9748-6865 (V.A. Markel)

elements of fluid in acoustics. This is different from the work done by the total field, which gives the absorbed power. An important point here is that the field inside any object is not equal to the incident field. For this reason, absorbed power is different from the extinguished power. We note that absorption is related to the work done by the actual physical field inside the object and therefore it can be related mathematically to the total inward energy flux through any surface enclosing the object. No such simple consideration exists for extinction.

The definitions of extinguished power in terms of volume integrals as in [13] or Fourier integrals as in [9] are not *operational*. This means that one can not access the physical quantities that enter these definitions by performing power measurements in free space. However, measurements of the latter type are typical in optics. Therefore, it is desirable to develop an operational definition of the extinguished power in terms of free-space energy fluxes. This problem was considered in a series of papers [14, 15, 16, 17, 18] for incident plane waves. A unifying goal of these references is to devise an experimental set-up in which the extinguished power can be measured with a single flat detector.

In [14], it was suggested that the power extinguished by a single particle can be measured by considering the energy flux into a circular detector placed behind the scatterer perpendicularly to the direction of incidence and subtracting the corresponding incident power, which would have been measured in the absence of the scatterer. It was found that, in order to obtain a converged result, the detector should subtend an angle of about $\pi/3$ when viewed from the scatterer. We reproduce this result below, although only for differential measurements and for some special positions of the measurement planes [1]. In [15], the results of Ref. [14] were generalized to the case of ordered or disordered arrangements of several particles, some of which can be located off the optical axis. It was found that the measurement aperture that is needed to achieve convergence is significantly smaller in this case.

Refs. [14, 15] were concerned only with the cross-term and the incident parts of the total energy flux. It was assumed that the scattered flux is small due to the $1/r^2$ dependence (we note that the paradox discussed below for collimated beams grows essentially from the same logic). However the flux that is quadratic in the scattered field is not always negligible. To obtain an accurate measurement of extinction, this (generally, unknown) flux should be subtracted from the power measured in the set up of [14, 15]. Otherwise, the measurements should be performed in such a way that the contribution of the scattered flux is in fact negligible. This problem was addressed in [16, 17, 18] as discussed in more detail below.

In [16], it was pointed out that, even for a single scattering particle, the integral of the cross-term in the energy flux converges much faster if the shape of the detector aperture is different from a circle centered on the axis of symmetry. A square detector was used in [16] to illustrate the point but similar results can be obtained even with a circular detector if it sufficiently displaced from the optical axis. This finding helps explain the result of [15], namely, that the cross-term integral converges faster for scatterers located off the optical axis. Moreover, it was shown in [16] that, for detectors located sufficiently far from the scatterer, convergence of the integral can be achieved while the contribution of the scattered power is still relatively small. This provides a direct operational definition of the extinguished energy, although, in order to obtained accurate results, certain conditions must be met. The detector should be small enough so that the scattered power can be neglected but large enough to average out the spurious oscillations in the intensity. This can be achieved if the scatterer is well in the Fraunhofer diffraction zone when viewed from the detector. We confirm these result below as well. The two notable differences between the simulations shown below and [16] are that we compute the current of energy rather than its density and, more importantly, we do not rely on the paraxial approximation and evaluate the relevant integrals numerically.

Finally, Refs. [17, 18] point out that the extinguished power can be accessed even not very far from the scatterer (i.e., not necessarily far in the Fraunhofer zone) by performing holographic (interferometric) measurements of the total field. Essentially, this entails multiple measurements with detectors of different size or one spatially-resolved measurement with a CCD camera located behind the scatterer. The incident flux is measured independently and then subtracted from the hologram in post-processing. However, the flux that is quadratic in the scattered field is unknown and can not be subtracted in this manner. In the earlier work [17], an approximation was adopted in which this unknown flux is not accounted for. In [18], this approximation was removed and it transpired that any integral measurement with a large area detector yields absorption rather than extinction (we confirm this result as well). To circumvent the problem, it was proposed to extract the extinguished power by post-processing the entire hologram. The post-processing entailed extrapolation to zero size (zero subtending angle) of the trend in the oscillatory relation between the registered power and the size of the detector. In this manner, accurate results were obtained for extinction from both simulated and experimental data [18]. Intuitively, this approach is consistent with the optical theorem, which contains the scattering amplitude only in the forward direction.

We note that the approaches of Ref. [16] and Refs. [17, 18] are closely related; both are aimed at removing or averaging out the spurious oscillations in the intensity. Ref. [16] relies on an intermediate asymptote in the dependence of the registered power on the detector size (when the latter is increased past the intermediate asymptote region, the mea-

[1] The differential measurements of the type described below are needed because the relevant integral taken over just one plane behind the scatterer diverges. The divergence occurs if a monochromatic scalar scattered wave contains a non-negligible isotropic component (the partial s-wave). For partially coherent waves or more complex apertures (including a square or circles displaced from the axis of symmetry), the divergence can be suppressed. This comment concerns only plane-wave illumination; for collimated beams, measurement on one plane is always sufficient.

surement crosses over to absorption rather than extinction). In Refs. [17, 18], this intermediate asymptote is extracted by post-processing. However, post-processing and extrapolating holographic data may not always be numerically stable. We will be interested therefore in a definition of extinction that utilizes integral power measurements only.

The rest of this article is organized as follows. In Section 2, we describe a simple scalar-wave model that is used in the paper. The case of incident plane waves is considered in detail in Section 3 and the case of an incident Gaussian beam in Section 4. Section 5 contains a brief discussion of the obtained results.

2. Model

For simplicity, we consider monochromatic scalar waves of the form $\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-i\omega t}$. The model is not completely general but encompasses some physical situations including quantum-mechanical and acoustic scattering. In the case of quantum mechanics, $\Psi(\mathbf{r},t)$ is the complex wave function whereas, in the acoustic case, the deviation of pressure from its equilibrium value is given by $\mathrm{Re}[\Psi(\mathbf{r},t)]$. The theory of this paper applies to both cases.

As is conventional in the scattering theory, we decompose the total field ψ into the incident and scattered components ψ_{i} and ψ_{s} so that

$$\psi(\mathbf{r}) = \psi_{\mathrm{i}}(\mathbf{r}) + \psi_{\mathrm{s}}(\mathbf{r}) \ . \tag{1}$$

We further assume that the scattering particle or potential is supported in a small region around the origin of a reference frame. Assuming this region fits entirely inside a ball of radius a, we require that $ka \ll 1$, where $k = \omega/c$ and c is the relevant phase velocity. Under these conditions, we can write

$$\psi_{\mathrm{s}}(\mathbf{r}) = \alpha\psi_{\mathrm{i}}(0)\frac{e^{ikr}}{r} \ , \quad k = \frac{\omega}{c} \ , \tag{2}$$

where r is the distance from the origin to the point of observation and α is a coefficient. Equation (2) follows from linearity of the underlying wave equation and smallness of the scatterer. Generalization of (2) to the case of multiple small scatterers is known as the Foldy-Lax approximation [19, 20].

For monochromatic fields, the stationary flux of energy (or probability) can be written as

$$\mathbf{j}(\mathbf{r}) = \mathrm{Im}[\psi^*(\mathbf{r})\nabla\psi(\mathbf{r})] \ . \tag{3}$$

The absorbed power Q_{a} is given by integrating the inward flux of energy over any (almost everywhere) regular surface $\mathbb{S}$ completely enclosing the origin, viz,

$$Q_{\mathrm{a}} = -\oint_{\mathbb{S}} \mathbf{j}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) d^2 r \ , \tag{4}$$

where $\hat{\mathbf{n}}(\mathbf{r})$ is the outward unit normal to $\mathbb{S}$ at $\mathbf{r} \in \mathbb{S}$. Using the decomposition (1), we can write $Q_{\mathrm{a}} = Q_{\mathrm{e}} - Q_{\mathrm{s}}$, where the extinguished and scattered powers Q_{e} and Q_{s} are given by

$$Q_{\mathrm{e}} = -\oint_{\mathbb{S}} \mathbf{j}_{\mathrm{e}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) d^2 r \ , \tag{5a}$$

$$Q_{\mathrm{s}} = \oint_{\mathbb{S}} \mathbf{j}_{\mathrm{s}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) d^2 r \ , \tag{5b}$$

and

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}) = \mathrm{Im}[\psi_{\mathrm{i}}^*(\mathbf{r})\nabla\psi_{\mathrm{s}}(\mathbf{r}) + \psi_{\mathrm{s}}^*(\mathbf{r})\nabla\psi_{\mathrm{i}}(\mathbf{r})] \ , \tag{6a}$$

$$\mathbf{j}_{\mathrm{s}}(\mathbf{r}) = \mathrm{Im}[\psi_{\mathrm{s}}^*(\mathbf{r})\nabla\psi_{\mathrm{s}}(\mathbf{r})] \ . \tag{6b}$$

Here we have accounted for the fact that the energy flux of the incident field integrates to zero over any closed surface that does not contain the source.

We thus see that only the absorbed energy is defined in terms of a *physical* energy flux, that is, a flux that actually exists in space. The scattered energy is defined in terms of the flux that is created by the scattered field alone. The extinguished power is defined in terms of a peculiar flux that is created by the interference between the incident and the scattered fields. This fact was previously noted as the main difficulty in deriving an operational definition of the extinguished power.

For the simple s-wave scattering that is described by (2), $\mathbf{j}_{\mathrm{s}}$ depends on the incident field trivially, as is given by

$$\mathbf{j}_{\mathrm{s}}(\mathbf{r}) = |\psi_{\mathrm{i}}(0)|^2 \frac{k|\alpha|^2}{r^2} \hat{\mathbf{r}} \ . \tag{7}$$

Integrating this expression over a sphere according to (5b), we obtain

$$Q_{\mathrm{s}} = 4\pi k |\psi_{\mathrm{i}}(0)|^2 |\alpha|^2 \ . \tag{8a}$$

The extinguished power can be obtained for an arbitrary incident field from the generalized optical theorem [13]. Specializing the more general result of [13] to the setting of this paper, we obtain

$$Q_{\mathrm{e}} = 4\pi |\psi_{\mathrm{i}}(0)|^2 \mathrm{Im}\alpha \ . \tag{8b}$$

Equation (8b) can be easily verified for an incident pane wave by integrating over a sphere. But it holds generally, as long as the incident field satisfies the underlying wave equation. We reiterate that simple relation of the form (8) are applicable to sufficiently small particles only.

We now see that the assumption of *passivity* of the scatterer places a constraint on α. Indeed, the condition that the scatterer does not generate energy reads $Q_{\mathrm{a}} = Q_{\mathrm{e}} - Q_{\mathrm{s}} \geq 0$. Using (8), we can transform this inequality to

$$\mathrm{Im}(1/k\alpha) \leq -1 \ . \tag{9}$$

Therefore, we can write

$$k\alpha = \frac{1}{x - iy} \ , \tag{10}$$

where x and y are real dimensionless variables and $y \geq 1$. We will say that the incident field is exactly in resonance with the scatterer if $x = 0$ and that the scatterer is non-absorbing if $y = 1$.

The measurement geometry considered in this paper is illustrated in Fig. 1. The scatterer is placed at the origin and its size and shape enter the problem only through the coefficient α. We then consider a cylindrical region Ω containing the origin. The surface of Ω is the union of two planar faces

$$\mathbb{S}_{\pm} = \{z = \pm L\ ;\ \ x^2 + y^2 \leq R^2\} \tag{11a}$$

and the cylindrical surface

$$\mathbb{D} = \{-L \leq z \leq L\ ;\ \ x^2 + y^2 = R^2\}\ . \tag{11b}$$

We will see that, under some conditions, the energy flux through $\mathbb{D}$ can be neglected.

We are interested in a measurement scheme utilizing one or two flat detectors, which integrate the electromagnetic power incident on their entire surfaces. We will place these detector in front and behind the scatterer on the surface $\mathbb{S}_{\pm}$. We will see that, in the case of plane-wave illumination and large-area detectors, measurements of this type yield absorption rather than scattering. Extracting extinction is still possible with R-resolved measurements and post-processing [17, 18] or by utilizing the intermediate asymptotic regime when the scatterer is far in the Fraunhofer zone of the detectors ($L \gg kR^2/2\pi$) [16]. However, in the case of incident Gaussian beams, extinction can be measured quite easily with only one flat detector located at an arbitrarily-selected plane behind the scatterer (within some range, of course), assuming the incident power is known or can be measured independently.

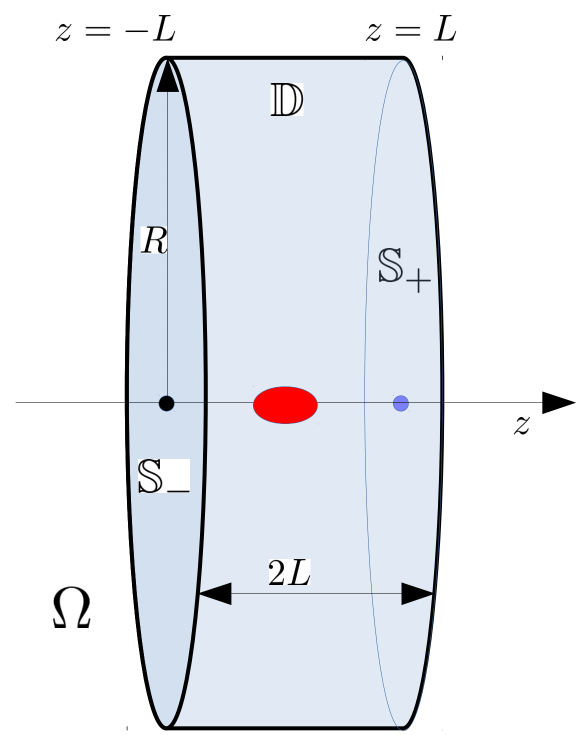

Figure 1: Illustration of the measurement surfaces that are considered in this paper. The radius of the base of the cylinder is R and the height is $2L$. Surface $\mathbb{S}_{\pm}$ and $\mathbb{D}$ are defined in (11).

3. Incident plane wave

Consider first the case when the incident field is a plane wave of the form

$$\psi_{\mathrm{i}} = Ae^{ikz}\ . \tag{12}$$

The current $\mathbf{j}_{\mathrm{s}}$ is then given by (7) with $\psi_i(0) = A$ and $\mathbf{j}_{\mathrm{e}}$ is given by

$$\mathbf{j}_{\mathrm{e}} = \frac{|A|^2}{r}\mathrm{Im}\left[\alpha e^{ik(r-z)}\frac{ikr-1}{r}\hat{\mathbf{r}} + ik\alpha^* e^{-ik(r-z)}\hat{\mathbf{z}}\right]\ . \tag{13}$$

Let us compute the power that enters the cylindrical region Ω shown in Fig. 1 through the planar surfaces $\mathbb{S}_-$ and $\mathbb{S}_+$. Let

$$\rho = \sqrt{x^2 + y^2} \tag{14}$$

be the distance from a point in $\mathbb{S}_{\pm}$ to the cylinder axis. We then define

$$\kappa_{\mathrm{e}}(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot [\mathbf{j}_{\mathrm{e}}(\rho, -L) - \mathbf{j}_{\mathrm{e}}(\rho, L)]\rho d\rho\ , \tag{15a}$$

$$\kappa_{\mathrm{s}}(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot [\mathbf{j}_{\mathrm{s}}(\rho, -L) - \mathbf{j}_{\mathrm{s}}(\rho, L)]\rho d\rho\ , \tag{15b}$$

$$\kappa_{\mathrm{t}}(R) = \kappa_{\mathrm{e}}(R) + \kappa_{\mathrm{s}}(R)\ . \tag{15c}$$

Thus, $\kappa_{\mathrm{t}}(R)$ is the power that enters Ω through $\mathbb{S}_-$ minus the power that leaves through $\mathbb{S}_+$. We will see momentarily that the power entering Ω through $\mathbb{D}$ can be neglected for some special values of L but not generally.

The conventional interpretation suggests that the extinguished power Q_{e} can be approximated by $\kappa_{\mathrm{t}}(R)$. One can hope that there exists an interval of R such that $\kappa_{\mathrm{s}}(R) \ll \kappa_{\mathrm{e}}(R)$ and $\kappa_{\mathrm{e}}(R) \approx Q_{\mathrm{e}}$. In this case, we can measure Q_{e} by measuring $\kappa_{\mathrm{t}}(R)$ in an appropriate interval of R. The important point here is that this definition of Q_{e} is operational since $\kappa_{\mathrm{t}}(R)$ is a physical energy flux, which is directly accessible in optical measurements.

Note that integrals of the individual terms $\hat{\mathbf{z}} \cdot \mathbf{j}_{\mathrm{e}}(\rho, \pm L)$ in (15a) do not have limits when $R \to \infty$. Integral of the difference of the $\pm$ terms also does not generally converge to a fixed limit except for some special values of L. Indeed, using (13), we can write

$$\kappa_{\mathrm{e}}(R) = 4\pi|A|^2\left[I_1(R)\cos(kL) + I_2(R)\sin(kL)\right]\ , \tag{16}$$

where

$$I_1(R) = \mathrm{Im}\left[\alpha\int_0^R \frac{\rho d\rho}{r^2}\left(\frac{L}{r} - ikL\right)e^{ikr}\right]\ , \tag{17a}$$

$$I_2(R) = \mathrm{Im}\left[k\alpha^*\int_0^R \frac{\rho d\rho}{r}e^{-ikr}\right] \tag{17b}$$

These integrals can be evaluated:

$$I_1(R) = \mathrm{Im}\left[\alpha\left(e^{ikL} - \frac{Le^{ik\sqrt{L^2+R^2}}}{\sqrt{L^2+R^2}}\right)\right]\ , \tag{18a}$$

$$I_2(R) = \mathrm{Re}\left[\alpha\left(e^{ik\sqrt{L^2+R^2}} - e^{ikL}\right)\right] . \tag{18b}$$

It can be seen that $I_2(R)$ does not have a limit when $R \to \infty$ but $I_1(R)$ has such a limit, although this limit depends on L.

There exists however an interesting special case. Namely, let us select L so that $\sin(kL) = 0$ or, equivalently, $e^{ikL} = \pm 1$. Then $I_2(R)$ does not enter (16) and the limiting value of $I_1(R)$ does not depend on L (as long as L takes one of the infinitely many discrete values for which $\sin(kL) = 0$). In this case, we obtain,

$$\kappa_{\rm e}(R) = 4\pi|A|^2 \mathrm{Im}\left[\alpha\left(1 \mp \frac{L e^{ik\sqrt{L^2+R^2}}}{\sqrt{L^2+R^2}}\right)\right] , \tag{19}$$

Here the "$-$" sign must be chosen if $e^{ikL} = 1$ and "$+$" if $e^{ikL} = -1$. But regardless of the sign, it can be seen that $\lim_{R\to\infty} \kappa_{\rm e}(R) = Q_{\rm e}$.

The above result is easy to understand. Under the condition $e^{ikL} = \pm 1$, the flux of energy through the cylindrical surface $\mathbb{D}$ approaches zero when $R \to \infty$. Indeed, the incident flux is directed along the Z-axis and is zero when projected onto the normal to $\mathbb{D}$ at any point. The flux associated with $\mathbf{j}_{\rm e}$ integrates to zero. We can use (6a) to see this easily. Indeed, the scattered field $\psi_{\rm s}(\mathbf{r})$ as well as its derivative $\nabla\psi_{\rm s}(\mathbf{r})$ are almost constant on $\mathbb{D}$ when $R \to \infty$. Therefore, the only spatial dependence in the integral that remains when $R \to \infty$ is due to the phase factor e^{ikz} in the incident wave. Integrating the exponent between z_1 and z_2 such that $e^{ikz_2} = e^{ikz_1}$ yields zero. In deriving (19), we have, essentially, required that this condition holds. Finally, it is easy to check that the flux of energy through $\mathbb{D}$ that is associated with $\mathbf{j}_{\rm s}$ vanishes as $1/R$.

Thus, it may seem that we can measure $Q_{\rm e}$ if we select L correctly and then integrate the energy flux over a sufficiently wide aperture of radius R. In practice, one would need to take $R \gg L$. This is inconvenient, of course, but more importantly this is not correct. The reason is that $\kappa_{\rm s}(R)$ is not at all negligible. A simple calculations shows that

$$\kappa_{\rm s}(R) = -4\pi k|\alpha|^2|A|^2\left(1 - \frac{L}{\sqrt{L^2+R^2}}\right) . \tag{20}$$

Combining (15c), (16) and (20), we conclude that

$$\lim_{R\to\infty} \kappa_{\rm t}(R) = 4\pi|A|^2\left(\mathrm{Im}\alpha - k|\alpha|^2\right) = Q_{\rm a} . \tag{21}$$

Thus, the limit of $\kappa_{\rm t}(R)$ when R goes to infinity is the absorbed, not the extinguished power. However, as discussed above, there exist some methods to extract $Q_{\rm e}$ from the measurements of $\kappa_{\rm t}$. This is discussed below in more detail.

The functions $\kappa_{\rm e}(R)$, $\kappa_{\rm s}(R)$ and $\kappa_{\rm t}(R)$ normalized to the theoretical value of the extinguished power, $Q_{\rm e}$, are plotted in Fig. 2 for a resonant non-absorbing scatterer with $k\alpha = i$ and various distances between the scatterer and the measurement planes as quantified by the parameter kL. The left (right) columns correspond to a circular (square) detectors, respectively. We note that in the case of a square detector, the volume Ω is not a cylinder such as the one shown in Fig. 1 but a cuboid. Since the integrals are not expressible in terms of elementary functions for the square aperture, we have evaluated them numerically.

Consider first the case when the parameter kL is relatively small and is a multiple of π, which is illustrated in Panels (a)-(d) of Fig. 2. Accounting for our choice to consider a non-absorbing particle with $Q_{\rm a} = 0$, we expect the curves shown in these panels to approach the following limits:

$$\lim_{R\to\infty} \kappa_{\rm e}(R)/Q_{\rm e} = \;\; 1 , \tag{22a}$$

$$\lim_{R\to\infty} \kappa_{\rm s}(R)/Q_{\rm e} = -1 , \tag{22b}$$

$$\lim_{R\to\infty} \kappa_{\rm t}(R)/Q_{\rm e} = \;\; 0 . \tag{22c}$$

This is indeed the case although the convergence is slow. At the aperture $R = 4L$, which implies a very wide solid angle of reception, the limits are still not reached. At small values of R, $\kappa_{\rm t}(R)$ is close to $\kappa_{\rm e}(R)$ (i.e., up to $R \approx 0.5L$), but the value of $\kappa_{\rm t}$ is still quite far away from $Q_{\rm e}$ for such small R. However, it can also be seen that $\kappa_{\rm e}(R)$ oscillates about its asymptotic value and the amplitude of oscillations is relatively smaller for the square detector. Moreover, in the case $kL = 40\pi$, the curve $\kappa_{\rm t}(R)$ (for the square detector) is not such a bad estimator of $Q_{\rm e}$ in the range $0.5 \lesssim R/L \lesssim 1.0$. Still, if only integrating detectors are used, measuring extinction with good accuracy is not possible if $kL = 10\pi$ or $kL = 40\pi$.

We next turn to the case $kL = 32.5$, which is not a multiple of π [Panels (e,f)]. In this case, the limits (22) are not reached for the circular aperture at all. This is in line with the observation that the energy that enters the cylindrical region Ω through its side surface $\mathbb{D}$ is zero only in the special cases $kL = \pi n$, where n is an integer. But, interestingly, the limits are still reached for the square aperture. In particular, the curve $\kappa_{\rm e}(R)$ in Panel (f) oscillates about $Q_{\rm e}$. The reason is that the volume Ω in the case of a square aperture is a cuboid rather than a cylinder. The energy flux through its four side faces is integrated to (almost) zero irrespective of kL. This is due to the property of square apertures to average out oscillations shaped as concentric circles. Thus, the use of a square detector removes the need to select special planes for measurement. Still, the parameter kL in Panels (e,f) is too small to neglect the contribution of $\kappa_{\rm s}$.

We next consider the parameter $kL = 1000\pi$ [Panels (g,h)]. For the circular aperture, the fluctuations are still very large and direct measurement of $Q_{\rm e}$ is not possible. However, for the square aperture [Fig. 2(h)], the oscillations die out at $R/L \gtrsim 0.25$. In this case, measurement of $Q_{\rm e}$ with about 5% to 10% relative error is possible by registering $\kappa_{\rm t}(R)$ at $R \approx 0.25L$. This is in agreement with the results of Ref. [16], namely, that square (more generally, not cylindrically-symmetric) detectors average out spurious oscillations in the intensity and thus make measurement of extinction with a single integrating detector possible. We have confirmed this conclusion by using direct numerical integration and not relying on the paraxial approximation.

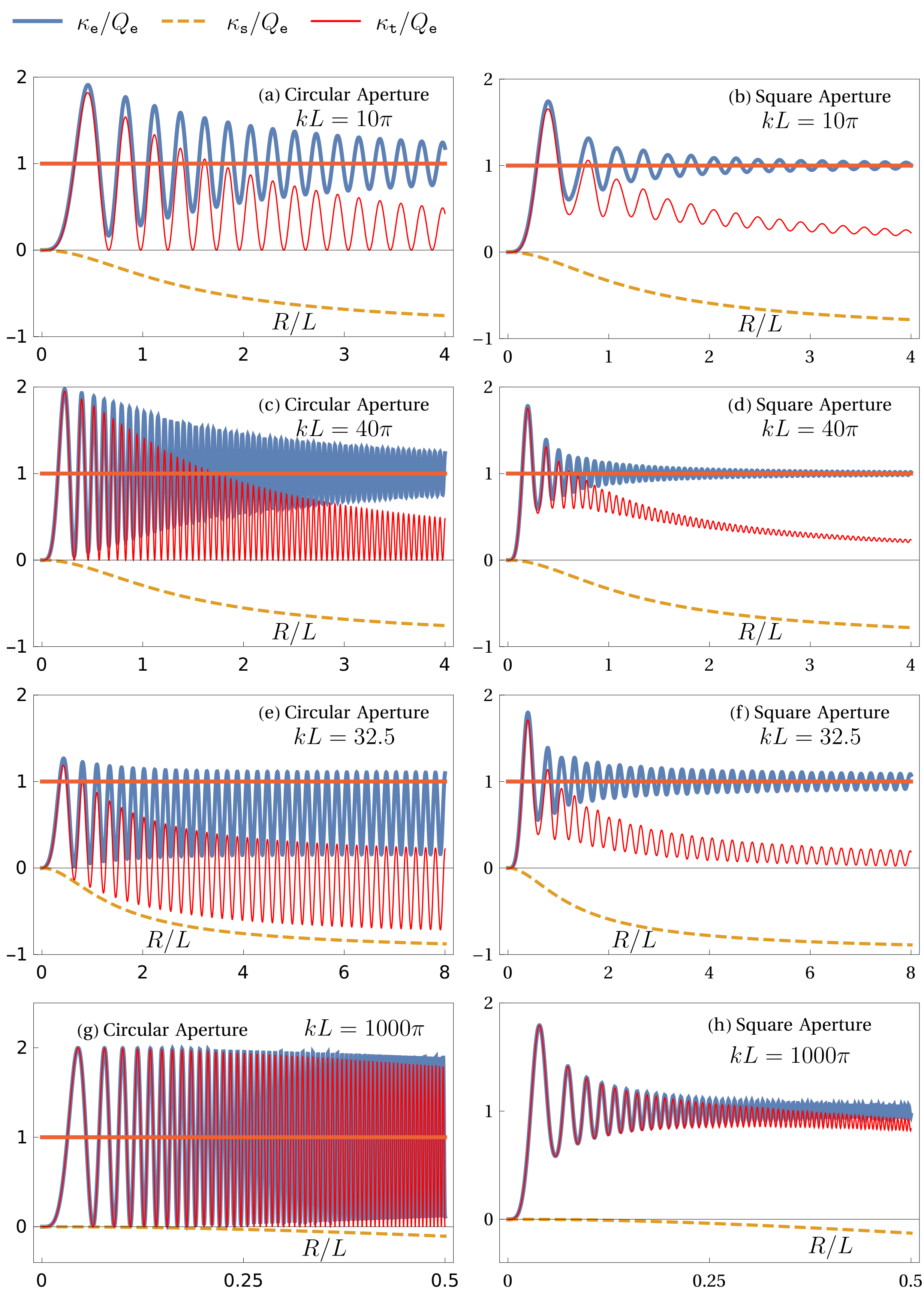

Figure 2: Functions $\kappa_e(R)$, $\kappa_s(R)$ and $\kappa_t(R)$ for $k\alpha = i$ and kL as labeled. Energy flux is integrated over a circular aperture of the radius R (left column – a,c,e,g) and over a square of side $2R$ (right column – b,d,f,h).

The above discussion concerned integrating detectors. However, the solid thin (red) curves in Fig. 2 are the radial profiles of the (integrated) holograms, which were studied in [18]. As was shown in [18], the data contained in these curves can be post-processed to yield $Q_{\rm e}$. The specific recipe given in [18] was to connect the minima point of $\kappa_{\rm t}(R)$ by a cubic spline, do the same for the maxima point, and then compute the trend as the average of these two splines. The trend is then evaluated at the argument corresponding to the first maximum of $\kappa_{\rm t}(R)$. A visual inspection of the solid thin curves in Fig. 2 suggests that this approach will indeed yield a result close to the true value of $Q_{\rm e}$. The extrapolation procedure relies on the observation that $Q_{\rm e}$ is the average of the first few oscillations of $\kappa_{\rm e}(R)$ taken over the range of R for which $\kappa_{\rm t}(R)$ is still close to $\kappa_{\rm e}(R)$. A coarser but probably a qualitatively similar estimate of $Q_{\rm e}$ can be obtained by taking one half of the first maximum of $\kappa_{\rm t}(R)$. We note however that for this prescription to work $\kappa_{\rm t}(R)$ should oscillate about $Q_{\rm e}$ at small R. This is not always the case for circular detectors.

4. Incident collimated beam

Interference effects in power flows for incident collimated beams are rarely considered in the literature. Two-dimensional beams incident perpendicularly to the axis of an infinite cylindrical scatterer have been considered in [3, 4] in relation to resolving the classical extinction paradox. However, an operational definition of extinguished power in terms of energy fluxes was not discussed in these works. Here we will perform a simple analysis of this kind and show that the conventional interpretation of extinguished power applies perfectly well to narrow collimated beams and small scatterers.

We start by noting that extinction of collimated beams involves an apparent paradox, which was discussed by us earlier [21]. Indeed, let us assume that the incident field is a perfectly collimated pencil beam as shown in Fig. 3. Then the area of the surface regions $\mathbb{S}_\pm$ where the incident and the scattered fields overlap does not depend on the distance to the scatterer, which scales as $\sim L$. We conclude that the scattered field on $\mathbb{S}_\pm$ decays as $\sim 1/L$. On the other hand, the incident field on $\mathbb{S}_\pm$ varies only due to the trivial factor $e^{\pm ikL}$ (since the beam is assumed to propagate without diffraction). Then the equations (5a) and (6a) seem to predict that $Q_{\rm e} \propto |A|^2 w^2/L$ where w is the beam radius. This scaling law is inconsistent with (8b). However, (8b) was derived in a mathematically rigorous way and should be correct.

Let us assume for the sake of the argument that the prediction $Q_{\rm e} \propto |A|^2 w^2/L$ is correct. Then, if we select the integration surface to be a sphere of radius L (which contains $\mathbb{S}_\pm$) and compute the total outward energy flux, we would obtain $Q_{\rm s}$ instead of $-Q_{\rm a}$. Strictly speaking, this integration yields $Q_{\rm s} - Q_{\rm e}$ but, if $Q_{\rm e}$ decays as $1/L$ and can be neglected for sufficiently large L, then the result is close to $Q_{\rm s}$. This conclusion contradicts conservation of energy: it suggests that the particle generates energy at the rate $Q_{\rm s}$.

It is tempting to resolve the above paradox by noting that a perfect pencil beam is a poor approximation. The radius

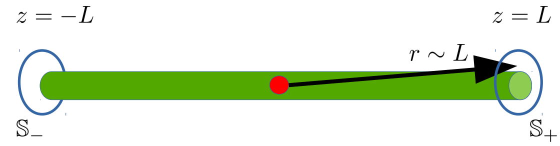

Figure 3: Illustration of the paradox involving extinction of a perfectly collimated beam by a small particle. The scattered field decays as $\sim 1/L$ on the surfaces $\mathbb{S}_\pm$ where the interference of the incident and the scattered fields occurs. It then follows that $Q_{\rm e} \sim w^2/L$, where w is the constant beam radius. This expression approaches zero when $L \to \infty$ in violation of energy conservation.

of a paraxial Gaussian beam, for example, increases linearly with L (for sufficiently large L) and for even larger propagation distances the paraxial approximation ceases to hold and the Gaussian beam behaves almost like a spherical wave. For these reasons, the original assumption that the area of $\mathbb{S}_\pm$ is independent of L is not generally correct. However, this observation is not sufficient to explain the paradox. Indeed, a Gaussian beam can be made arbitrarily slowly diverging in an arbitrarily large (but not infinite, of course) range of propagation distances. So, while the above effects are real, their onset does not occur fast enough to resolve the paradox. Below, we will show that the resolution is quite different and does not require or imply diffraction of the incident field or dependence of the area of $\mathbb{S}_\pm$ on L.

Let the incident field be given by a cylindrically-symmetric scalar Gaussian beam. The exact expression for this field is

$$\psi_{\rm i}(\rho, z) = A\frac{2}{\sigma^2}\int e^{-(q/\sigma k)^2} e^{ikz\sqrt{1-(q/k)^2}} J_0(q\rho) q dq \, , \quad (23)$$

where σ is the dimensionless parameter determining the beam waist, $J_0(x)$ is the Bessel function of the first kind and A is the amplitude. We note that, as before, $\psi_{\rm i}(0) = A$.

The expression in the right-hand side of (23) satisfies the wave equation exactly. However, if z is sufficiently small, we can make the paraxial approximation by expanding the square root as

$$\sqrt{1-\left(\frac{q}{k}\right)^2} = 1 - \frac{1}{2}\left(\frac{q}{k}\right)^2 - \frac{1}{8}\left(\frac{q}{k}\right)^4 + \dots$$

and keeping only the first two terms in this expansion. It can be seen that the condition of applicability of the paraxial approximation is $kz \ll 4\pi/\sigma^4$. For small σ, the critical distance at which the paraxial approximation breaks can be very large. If we keep, as suggested, only the first two terms in the above expansion, the integral (23) can be evaluated analytically with the result

$$\psi_{\rm i}(\rho, z) = \frac{A}{1 + i\sigma^2 kz/2} \exp\left[ikz - \frac{(\sigma k\rho/2)^2}{1 + i\sigma^2 kz/2}\right] \, , \quad (24)$$

The beam width in the paraxial approximation is given by the well-known formula

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2} \, , \quad w_0 = \frac{2}{\sigma k} \, , \quad z_0 = \frac{2}{\sigma^2 k} \, . \quad (25)$$

We thus see that there exists a range of z such that $2/\sigma \ll kz \ll 2/\sigma^2 \ll 4\pi/\sigma^4$ in which the paraxial approximation is accurate, the propagation distance is much larger than the beam waist, and there is still no noticeable diffraction. For example, if we take $\sigma = 0.001$ (in optics, this choice corresponds to a beam of about 1mm in radius at the waist), the range of z in which the beam is non-diffracting covers about two decades. In principle, one can consider even smaller values of σ. This observation clearly indicates that diffraction of the incident beam is not sufficient to resolve the paradox described above.

Let us consider again the functions $\kappa_{\rm e}(R)$, $\kappa_{\rm s}(R)$ and $\kappa_{\rm t}(R)$ that were defined in (15). The integrands in (15) depend on the fluxes $\mathbf{j}_{\rm e}(\rho, \pm L)$, which are expressed in terms of the incident and scattered fields in (6a). In Section 3, we have used the plane wave (12) for the incident field; now we will use the paraxial Gaussian beam (24). Note that the scattered field is the same in both cases and given by (2) with $\psi_{\rm i}(0) = A$. With this in mind, let us rewrite (15a) as

$$\kappa_{\rm e}(R) = \frac{2\pi|A|^2}{L}\int_0^R f_{\rm e}(\rho, L)\rho d\rho \,, \tag{26}$$

where

$$f_{\rm e}(\rho, L) = \frac{L}{|A|^2}\hat{\mathbf{z}}\cdot[\mathbf{j}_{\rm e}(\rho,-L) - \mathbf{j}_{\rm e}(\rho, L)] \,. \tag{27}$$

Note that $f_{\rm e}$ is dimensionless. In Fig. 4(a), we plot $f_{\rm e}(\rho, L)$ as a function of ρ for $\sigma = 0.001$ and $k\alpha = i$ (as above). It can be seen that $f_{\rm e}(\rho, L)$ is of the same order of magnitude for the different values of kL. It might seem therefore that (26) predicts that $\kappa_{\rm e}(R) \sim 1/L$. Recall that, on one hand, $\lim_{R\to\infty}\kappa_{\rm e}(R) = Q_{\rm e}$ but, on the other hand, we know theoretically that $Q_{\rm e}$ is given by (8b), which is independent of L. The explanation of this apparent contradiction is as follows. The rate of oscillation of $f_{\rm e}(\rho, L)$, when considered as a function of ρ, tends to decrease with L. As a result, the integral in (26) is approximately proportional to L. The proportionality becomes more precise when R is increased. Therefore, the factor of L in (26) is effectively canceled. This is illustrated in Fig. 4(b) where we show that $\kappa_{\rm e}(R)$ approaches its theoretical limit (8b) irrespective of L. Convergence is achieved when $R/w_0 \gtrsim 3$. For smaller L, $\kappa_{\rm e}(R)$ makes more oscillations before reaching its limiting value.

Thus, the resolution of the paradox turns out to be very simple. Even though the scattered field decreases as $1/L$ for $\mathbf{r} \in \mathbb{S}_\pm$, the total energy flux associated with $\mathbf{j}_{\rm e}$ remains independent of L as long as integration in (26) is carried out to sufficiently large values of R. This happens because the rate of oscillation of the integrand $f_{\rm e}(\rho, L)$ tends to decrease with L. In other words, the paradox was based on an implicit assumption that $Q_{\rm e}$ is determined only by the "interaction area" and the amplitude of the integrand. However, this is not so for oscillatory integrals because the phase and the frequency of the integrand oscillations also matter. A similar mathematical behavior was observed in the case of plane waves [14, 16].

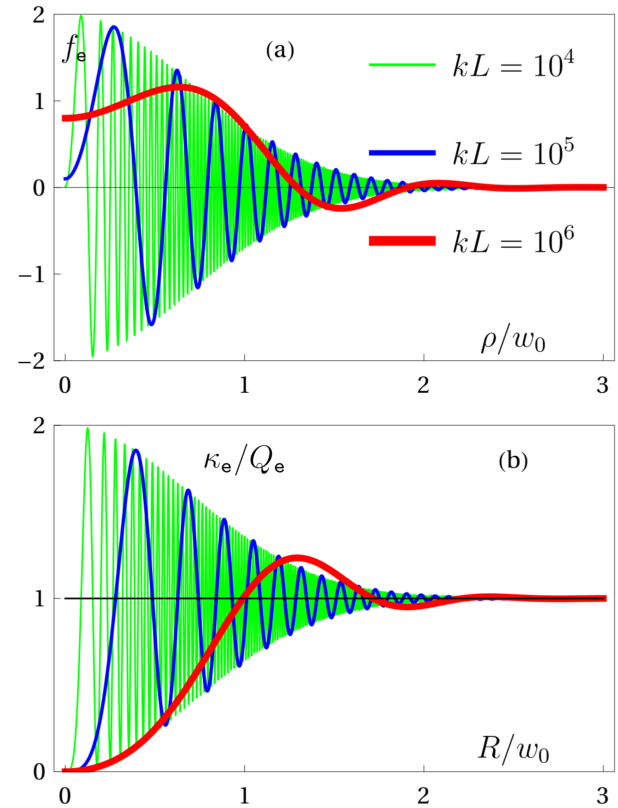

Figure 4: (a) Function $f_{\rm e}(\rho, L)$ defined in (27) for an incident paraxial Gaussian beam with $\sigma = 0.001$, $k\alpha = i$ and different values of kL, as labeled; (b) $\kappa_{\rm e}(R, L)$ normalized to its theoretical limit $Q_{\rm e}$ and computed by numerical integration of $f_{\rm e}(\rho, L)$ according to (26) for the same values of kL as in Panel (a). The variable ρ is shown in units of the waist radius $w_0 = 2/\sigma$.

The above consideration is applicable for $2/\sigma \ll kL \ll 2/\sigma^2$, that is, while diffraction of the Gaussian beam is not significant. The numerical values of kL used in Fig. 4 are in this range, although the inequality is not particularly strong for $kL = 10^6$. We now discuss briefly what happens outside of the above interval of L. Of course, oscillations of $f_{\rm e}(\rho, L)$ can not get slower indefinitely. When $kL = 2/\sigma^2$, $f_{\rm e}(\rho, L)$ is almost constant on the scale of $\sim w_0$. However, the area of $\mathbb{S}_\pm$ starts to increase with L for $kL \gtrsim 2/\sigma^2$. This yields the theoretically expected result $\lim_{R\to\infty}\kappa_{\rm e}(R) = Q_{\rm e}$ although integration now should be carried out to larger values of R. We do not show the corresponding plots as they do not reveal anything conceptually new. This behavior is also easily understandable gometrically.

For even larger values of L such that $kL \gtrsim 4\pi/\sigma^4$, the paraxial approximation becomes inaccurate and one must use the exact expression (23) for the incident field. Numerical integration becomes difficult in this case but nothing unexpected happens. Since (23) satisfies the wave equation exactly, the generalized optical theorem also holds exactly; it is just necessary to integrate $f_{\rm e}(\rho, L)$ over an increasingly large area.

In order to provide an operational method of measuring

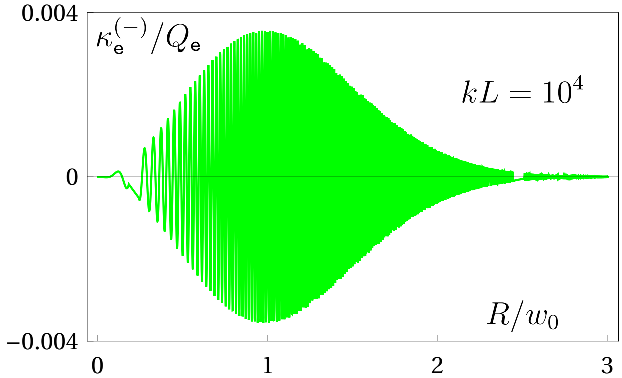

Figure 5: Function $\kappa_{\rm e}^{(-)}(R)$ for the same parameters as in Fig. 4 but only the case $kL = 10^4$ is shown. The numerical values of $\kappa_{\rm e}^{(-)}(R)$ for $kL = 10^5$ and $kL = 10^6$ are too small to be shown in the same figure frame.

$Q_{\rm e}$, it remains to show that $\kappa_{\rm s} \ll \kappa_{\rm e}$ for some reasonable measurement parameters. This relation could be expected since $\mathbb{S}_\pm$ typically make a small fraction of the sphere of radius L (in terms of area) and scattering by small particles is isotropic or, at least, not very forward-peaked. We can make the statement more quantitative for the parameters used in Fig. (4). Consider $R = 2w_0$ as the characteristic point. Then $|\kappa_{\rm s}/\kappa_{\rm e}| \approx 0.07$ for $kL = 10^4$, $|\kappa_{\rm s}/\kappa_{\rm e}| \approx 8\ 10^{-4}$ for $kL = 10^5$ and $\approx 8\ 10^{-6}$ for $kL = 10^6$. So, in the cases considered above, the scattered and incident fields can be effectively separated and the extinguished power can be measured in terms of energy fluxes.

Note that $\kappa_{\rm e}$ was defined in (15a) using the difference between the flux of $\mathbf{j}_{\rm e}$ that enters the region Ω through $\mathbb{S}_-$ minus the flux that exits through $\mathbb{S}_+$. However, the first quantity can be negligible for some "useful" values of R. By useful, we mean such R that $\kappa_{\rm e}$ is already close to its limiting value yet $\kappa_{\rm s}$ is still negligible. This is illustrated in Fig. 5 where we plot the function

$$\kappa_{\rm e}^{(-)}(R) = 2\pi \int_0^R \hat{\mathbf{z}} \cdot \mathbf{j}_{\rm e}(\rho, -L)\rho d\rho \tag{28}$$

for the same parameters as in Fig. 4 and $kL = 10^4$. For the larger values of L that were considered in Fig. 4, $\kappa_{\rm e}^{(-)}(R)$ is even smaller. This is due to cancellation of terms in (6a). The two terms in the right-hand side of this expression interfere destructively at $z = -L$. In addition, $\kappa_{\rm e}^{(-)}(R)$ approaches its limiting value of zero when $R \sim 3w_0$.

Finally, we should not forget about the incident energy flux. What we have shown so far is that the energy fluxes $Q_{\rm e}^{(-)}$ and $Q_{\rm s}^{(-)}$ through $\mathbb{S}_-$ are negligible. However, the incident energy flux is just the incident power, W. We therefore have $Q_{\rm t}^{(-)} \approx W$ for the total energy flux through $\mathbb{S}_-$. Analogously, we have $Q_{\rm t}^{(+)} \approx W - Q_{\rm e}$. The approximate equality is used to indicate that we disregarded various negligible terms discussed above. The difference, $\Delta Q \equiv Q_{\rm t}^{(-)} - Q_{\rm t}^{(+)}$ is therefore a good approximation for $Q_{\rm e}$. However, if the incident energy flux W is known *a priori*, then $Q_{\rm e}$ can be measured with a single flat detector located on $\mathbb{S}_+$. In this case, the measurement scheme becomes identical to that proposed in Ref. [18].

Thus, we have an operation definition of extinction in terms of directly measurable energy fluxes. Importantly, the extinguished energy $Q_{\rm e}$ can be measured with just one flat, power-integrating detector, assuming the incident power is known or can be measured independently. The shape and size of the detector does not matter as long as it is large enough to intercept the beam but small enough to not be affected by the scattered energy flux. Under these conditions, $Q_{\rm s}$ can also be measured directly by using 4π-solid angle detectors, although it is probably much easier to measure $Q_{\rm e}$ and $Q_{\rm a}$ with flat detectors of different size (a small detector will measure $Q_{\rm e}$ and two large detectors would still measure $Q_{\rm a}$ as in the previous Section) and then compute the scattered power as $Q_{\rm e} - Q_{\rm a}$.

5. Discussion

We have shown that extinguished power is a meaningful physical quantity for narrow collimated beams and small, almost isotropically scattering particles. In this regime, the extinguished power is literally removed from the beam. However, as we relax the above assumptions, definition of extinguished power in terms of directly measurable energy fluxes becomes not impossible but increasingly problematic.

In particular, the extinction cross section does not characterize fully small particles for the case of a very widefront incident waves. This is obvious already from the following example: a periodic arrangement of non-absorbing small particles support a wave that propagates through such a medium without any attenuation even though each particle can have a large extinction cross section. If we then consider a disordered arrangement, all incident power will be backreflected (for a sufficiently thick layer of such particles), but the rate of decay of the wave inside the layer will depend on the scattering phase function of each particle, not on its total extinction cross section. This is well-known in the theory of radiative transport.

Another limitation is related to strongly forward-peaked scattering. Large objects create a geometric shadow behind them and this is related to a highly forward-peaked scattering amplitude. Measuring the extinguished power in this case is problematic because it is difficult to separate spatially the incident and the scattered fields. Basically, the larger the scattering object is, the further from it the measurement surfaces must be placed. On the other hand, we need to make sure that the incident beam is still tightly collimated at such propagation distances. The two requirements contradict each other. Therefore, although it is theoretically possible to construct a collimated beam so that the power extinguished by any finite object can be measured in the manner described above, the parameters of this beam can easily become extreme or unrealistic.

We have considered extinction by a single isolated particle. Additional complicated effects arise when the particle can move, or there are more than one particle, or there are several particles that can move. The above effects can make measuring the extinction easier by averaging out the oscillations in the interference term [15]. On the other hand, multiple-scattering effects can make the theoretical considerations more complicated, as can be glimpsed from the above example of a thick layer containing many particles.

Scalar Gaussian beams have a relatively simple mathematical form, and this was the main motivation for considering a scalar field in this paper. We expect however that the same conclusions can be reached in the more complicated electromagnetic case, although some new polarization-related effects will arise, as is demonstrated in [10]. A linearly-polarized Gaussian beam is not very different from the scalar beam considered below. However, one important difference between the scalar wave and vector wave physics is that, in the latter case, the scattered field is never truly isotropic. The effects of the scattered field anisotropy on the operational definition of extinction is an interesting subject for further investigation.

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