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On the current-driven model in the classical electrodynamics of continuous media

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Abstract
The current-driven model in which a continuous medium is excited by a pre-determined current which overlaps with the medium in all points in space but is not subject to constitutive relations is critically analyzed.

1. Introduction

Recently, significant attention has been drawn to the theory of electromagnetic homogenization of periodic media [1–4]. The interest is motivated by the proposals to create artificial electromagnetic materials (metamaterials) with exotic properties not encountered in Nature. Suggested applications of the metamaterials range from high-resolution microscopy [5] to creating novel efficient antennas [6]. In all cases I am aware of, the homogenization theories which are considered in the metamaterial community are, essentially, macroscopic. That is, one starts from the macroscopic Maxwell's equations for a nonmagnetic periodic medium characterized by a spatially varying permittivity $\epsilon(\mathbf{r})$ (and $\mu(\mathbf{r}) = 1$) and seeks to approximate it by a medium of the same overall shape but characterized by spatially uniform effective parameters $\epsilon_{\rm eff}$ and $\mu_{\rm eff} \neq 1$. It has been suggested that a mixture of two intrinsically nonmagnetic substances can have a nontrivial magnetic response at certain resonance frequencies [7].

In spite of a large body of literature published on the subject, the theory of homogenization continues to attract attention [8]. Recently, a new approach to homogenization has emerged which is based on the so-called current-driven model and is exemplified by [4, 9]. The current-driven model is deeply rooted in the theory of low-frequency electromagnetic devices. When such devices are considered, it is customary to view the electric current running in one or several wires or antennas as the source of electromagnetic fields in the surrounding space. This source current is frequently referred to as the free or the external current. When Maxwell's equations are solved, it is assumed that the external current is fully controlled by the experimentalist. The problem then consists in finding the distribution of electromagnetic fields and induced currents everywhere outside of the wires which carry the pre-determined external current. For instance, the current in a receiver antenna is considered to be induced; it is not directly controlled by the experimentalist and must be determined by solving the appropriate scattering problem.

The above approach is physically and mathematically sound and has been used in electrical engineering with great success. However, in [4, 9], the current-driven model is used far outside of its area of applicability. In particular, it is assumed that the external current is a plane wave of infinite extent which directly overlaps with a macroscopically large sample of a continuous medium. The overlap occurs not just along several wires (which one can hope to insert into the medium) but in the whole space. Moreover, the wavevector of this plane wave is viewed as a mathematically independent variable which is unrelated to the medium properties. Moreover, the medium in question is assumed to be unbounded and of infinite extent. Apart from the fact that such media do not exist in Nature, the electromagnetic processes which occur at a medium boundary are important and should not be left out of consideration. More specifically, every medium, in addition to a dispersion relation and a refractive index, is also characterized by an impedance.

The present paper contains a critical analysis of the current-driven model of [4, 9]. Such analysis is particularly important because the model is being promoted not only as a means to homogenization but also as a general, first principle approach to solving electromagnetic problems.

Fundamentally, there are two questions to ponder. The first question is whether the excitation model of [4, 9] is physically realizable in principle. The second question is the following: even if the external current of the form proposed in [4, 9] cannot be realized experimentally, would introduction of such a current serve, perhaps, as a convenient mathematical tool for computing certain physically measurable quantities? My answer to both questions is no. Correspondingly, I find that the physical conclusions that have been drawn from the current-driven model are all incorrect.

2. Mathematical formulation of the current-driven model

In the current-driven model of [4, 9], it is assumed that the electromagnetic fields in a passive medium (either spatially uniform or not) are excited by an external current which overlaps with the medium but is not subject to constitutive relations. This external current appears as a source term in the macroscopic Maxwell's equations, namely,

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\rm e}, \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{I}_{\rm e}, \end{aligned} \tag{1}$$

where the Gaussian units have been used. In (1), $\mathbf{J}_{\rm e}$ and $\mathbf{I}_{\rm e}$ are the external currents of electric and magnetic charges, respectively. In [9], only the electric current $\mathbf{J}_{\rm e}$ was used while in [4] both currents were used. It should be clarified that the physical existence of magnetic monopoles was not assumed in [4]; the magnetic current was introduced only as a mathematical manipulation.

It is further accepted that the medium also supports the induced electric current $\mathbf{J}_{\rm d} = \partial \mathbf{P}/\partial t + c\nabla \times \mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the vector fields of polarization and magnetization. In the conventional electrodynamics, only the induced current of electric charge exists. For this reason, the field $\mathbf{I}_{\rm d}$ is not introduced. Unlike the external current, the induced current obeys the constitutive relations. With the usual definitions $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, the system of equations (1) can be equivalently re-written as

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}(\mathbf{J}_{\rm d} + \mathbf{J}_{\rm e}), \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{I}_{\rm e}. \end{aligned} \tag{2}$$

Equations (1) or (2) form a perfectly valid mathematical formulation of the electromagnetic problem as long as the spatial support of the functions $\mathbf{J}_{\rm e}(\mathbf{r}, t)$ and $\mathbf{I}_{\rm e}(\mathbf{r}, t)$ does not overlap with the continuous medium. If the above condition is satisfied, the terms $\mathbf{J}_{\rm e}(\mathbf{r}, t)$ and $\mathbf{I}_{\rm e}(\mathbf{r}, t)$ describe external sources of radiation which can be directly controlled by the experimentalist. This approach, although valid, is rarely used for solving electromagnetic boundary-value problems involving reflection and refraction of waves in bulk samples. For example, in the case of laser irradiation, it is more convenient to define the source of the electromagnetic fields as an incident wave whose properties are known and well characterized rather than by complicated currents inside the laser. Still, these two approaches are mathematically equivalent.

However, in [4, 9], the external currents have been taken in the form of plane waves:

$$\begin{aligned} \mathbf{J}_{\rm e}(\mathbf{r}, t) &= {\rm Re}[\mathbf{J}_0 \exp({\rm i}\mathbf{k}\cdot\mathbf{r} - {\rm i}\omega t)], \\ \mathbf{I}_{\rm e}(\mathbf{r}, t) &= {\rm Re}[\mathbf{I}_0 \exp({\rm i}\mathbf{k}\cdot\mathbf{r} - {\rm i}\omega t)]. \end{aligned} \tag{3}$$

The support of the functions defined in (3) is infinite and necessarily overlaps with the medium. In section 3, we will consider the question of whether this excitation model is physically realizable.

3. Is the current-driven model physical?

The authors of [4, 9] have expressed the opinion that the external current of the form (3) can be physically realized in some experiments. Thus, Fietz and Shvets suggested that such experiments may be encountered in 'applications such as novel antennas embedded in metamaterial shells' [4]. Similarly, Silveirinha has stated that it might be challenging but not impossible to place a generator within each unit cell of a periodic composite medium to create the external current of the form (3) with an arbitrary wavevector $\mathbf{k}$ [10].

In reality, it is not challenging to create an external source current of the type (3) but plainly impossible. Any generator running a pre-determined current must occupy a finite volume of space which can no longer be considered as part of the medium. Inside this volume, constitutive relations characteristic of the medium no longer hold. The volume is enclosed by a surface at which the usual Maxwell's boundary conditions are applicable. Thus, Silveirinha proposes, essentially, to take out a piece of the medium and to place an externally controlled current generator in the created void. This is illustrated in figure 1. Of course, in both cases shown in this figure, the source cannot be considered as part of the medium and the usual Maxwell's boundary conditions must be applied at all interfaces. In the case B, the boundary conditions must be applied not only at the outer boundary of the medium, but also at the surface enclosing the source. Further, there must be many such generators in order to mimic a smooth plane wave. Therefore, in order to create the external current of the form (3), the whole medium must be replaced by the 'excluded volumes' such as the one shown in figure 1(B). Doing so would surely destroy the medium. Here I do not even raise the question of how the proposed current generators would be supplied with energy.

A similar consideration applies to a 'novel antenna' surrounded by a metamaterial nanoshell. The current running in the antenna is drastically different from an infinite smooth plane wave and the material of the antenna cannot overlap with the metamaterial shell.

From a purely practical point of view, it should suffice to say that the excitation scheme with multiple current generators placed inside the medium is simply not the way experiments are done. Therefore, this exotic excitation scheme is

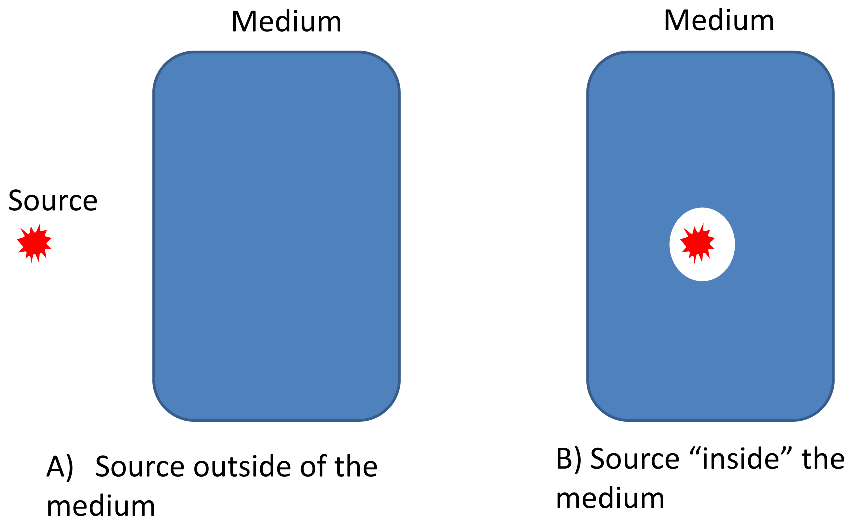

Figure 1. Illustration of the concept of placing a source 'inside' a continuous medium. In reality, in both cases shown, the source occupies a region of space which is not part of the medium. Maxwell's boundary conditions must be applied on all surfaces of discontinuity (the blue–white interfaces).

(This figure is in colour only in the electronic version)

introduced not to describe the experiments more faithfully but to simplify analytical derivations. Indeed, the current-driven model effectively replaces an eigenproblem of determining the complex Bloch wavevector and the complex impedance of a periodic medium by the much simpler problem of driven oscillations in an unbounded medium. However, the overriding consideration of any physical theory is not convenience but correctness. In section 4, we will consider the soundness of the current-driven model from a more mathematical point of view.

4. Is the current-driven model a useful mathematical tool?

Even if one accepts that the excitation model described by equation (3) cannot be realized experimentally, there is still a possibility that the introduction of this excitation scheme is a convenient mathematical manipulation which can be used to compute certain measurable quantities. In support of this proposition, Silveirinha has noted that any localized monochromatic current whose support does not overlap with the medium (which is an experimentally realizable excitation scheme) can be expanded into the spatial Fourier integral and, therefore, represented by a superposition of plane waves with real wavevectors. From this, Silveirinha concludes that 'the response to a plane-wave-like excitation is full of physical meaning, and even if such external sources may be challenging to realize in practice, the response to any given practical source may be obtained from the response of the material to plane-wave-like excitations' [10].

The statements quoted above can be rebutted by noting the following.

(i) Even though all fields and currents can be expanded into Fourier integrals, the individual modes in these expansions cannot be mathematically related to measurable quantities. All computations must be done for the actual field which exists in the medium.
(ii) Boundary-value problems in the electromagnetic theory cannot be solved or reduced to quadratures by spatial Fourier transform because finite objects are not translationally invariant.
(iii) Physical quantities which are quadratic in the fields do not satisfy the superposition principle and, therefore, cannot be computed by summing up the contributions of individual modes. In other words, interference effects must be taken into account.

We now discuss these points and their relation to the current-driven model of [4, 9] in more detail.

4.1. A toy-problem example

The claim that an individual mode in the Fourier expansion of various fields can be invested with an independent physical meaning is conceptually similar to the following mathematical fallacy. Consider a set of functions $f(x)$ defined in the interval $x \in (0, \infty)$, and let a certain physical quantity $\mathcal{P}(x)$ be determined by the formula $\mathcal{P}(x) = -f^{-1}(x)\,\mathrm{d}f(x)/\mathrm{d}x$. The correct form of $f(x)$ must be obtained by solving some equations which are not stated here. Assume now that we have solved these equations and have found that $f(x) = \exp(-px)$, where p is a positive constant. From this, we find that $\mathcal{P}(x) = p$. Now let us compute $\mathcal{P}(x)$ differently. Namely, let us first expand the function $\exp(-px)$ into a Fourier integral by writing

$$\exp(-px) = \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}kx)}{p + \mathrm{i}k} \frac{\mathrm{d}k}{2\pi}, \qquad x > 0. \tag{4}$$

We now compute $\mathcal{P}(x)$ for one of the plane-wave modes used in the above expansion and find that $\mathcal{P}(x) = -\mathrm{i}k$, where k is real and otherwise arbitrary. This result is obviously incorrect.

Similarly, even though any field propagating in an absorbing medium can be expanded into a basis of non-decaying plane waves, it is incorrect to compute any measurable quantity by retaining only one of the expansion modes. The computation must be done for the actual field which exists in the medium. Some quantities obey the superposition principle and others do not; however, none can be computed correctly by using a single arbitrarily chosen mode.

4.2. Maxwell's equations by the spatial Fourier transform

Consider a spatially uniform nonmagnetic medium which occupies the region V. The source of all electromagnetic fields is the external current $\mathbf{J}_e$ whose support does not overlap with V. All fields and currents are monochromatic so that we can write

$$\mathbf{J}_e(\mathbf{r}, t) = \mathrm{Re}[\mathbf{J}_e(\mathbf{r}) \exp(-\mathrm{i}\omega t)], \tag{5}$$

and similarly for other fields. Now let us expand, as was suggested by Silveirinha, $\mathbf{J}_e(\mathbf{r})$ into the Fourier integral. The forward and inverse expansions read

$$\begin{aligned} \tilde{\mathbf{J}}_e(\mathbf{k}) &= \int \mathbf{J}_e(\mathbf{r}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r})\, \mathrm{d}^3 r, \\ \mathbf{J}_e(\mathbf{r}) &= \int \tilde{\mathbf{J}}_e(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \frac{\mathrm{d}^3 k}{(2\pi)^3}. \end{aligned} \tag{6}$$

Here the tilde denotes the Fourier transform and similar expansions can be written for the electric field $\mathbf{E}$, the displacement $\mathbf{D}$, and for all other fields. Note that the integrals in (6) are evaluated over the whole space.

The next step is to state the Maxwell's equations. Since we have assumed that the medium is nonmagnetic, $\mathbf{B} = \mathbf{H}$. In this case, the Maxwell's equations in the frequency domain read

$$\begin{aligned} \nabla \times \mathbf{H}(\mathbf{r}) &= -\mathrm{i}\frac{\omega}{c}\mathbf{D}(\mathbf{r}) + \frac{4\pi}{c}\mathbf{J}_e(\mathbf{r}), \\ \nabla \times \mathbf{E}(\mathbf{r}) &= \mathrm{i}\frac{\omega}{c}\mathbf{H}(\mathbf{r}). \end{aligned} \tag{7}$$

We can exclude the field $\mathbf{H}$ from the above equations to obtain

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{D}(\mathbf{r}) + \frac{4\pi \mathrm{i}\omega}{c^2}\mathbf{J}_e(\mathbf{r}). \tag{8}$$

We now substitute the expansion (6) and similar expansions for $\mathbf{E}$ and $\mathbf{D}$ into (8). This results in

$$-\mathbf{k} \times \mathbf{k} \times \tilde{\mathbf{E}}(\mathbf{k}) = \left(\frac{\omega}{c}\right)^2 \tilde{\mathbf{D}}(\mathbf{k}) + \frac{4\pi \mathrm{i}\omega}{c^2}\tilde{\mathbf{J}}_e(\mathbf{k}). \tag{9}$$

To proceed, we need to relate $\tilde{\mathbf{D}}(\mathbf{k})$ to $\tilde{\mathbf{E}}(\mathbf{k})$. If a simple linear relation between $\tilde{\mathbf{D}}(\mathbf{k})$ to $\tilde{\mathbf{E}}(\mathbf{k})$ could be established, then equation (9) would be reduced to an analytically solvable algebraic equation. The real-space solution would then be obtainable by the inverse Fourier transform and, thus, the solution to an arbitrary boundary-value problem would be reduced to a quadrature. This is, of course, too good to be true. In reality, a simple linear proportionality between $\tilde{\mathbf{D}}(\mathbf{k})$ and $\tilde{\mathbf{E}}(\mathbf{k})$ results only in infinite unbounded media; in finite samples, the proportionality is replaced by a more general integral transform. Indeed, consider the simple case of a local dielectric response given by the function

$$\epsilon(\omega, \mathbf{r}) = \begin{cases} \epsilon_m(\omega) \neq 1, & \text{if } \mathbf{r} \in V \\ 1, & \text{if } \mathbf{r} \notin V. \end{cases} \tag{10}$$

Here $\epsilon_m(\omega) \neq 1$ is the permittivity of the medium at the working frequency. From the definition of the displacement, $\mathbf{D}(\mathbf{r}) = \epsilon(\omega, \mathbf{r})\mathbf{E}(\mathbf{r})$, and from the Fourier transformation rules, we find that

$$\tilde{\mathbf{D}}(\mathbf{k}) = \tilde{\mathbf{E}}(\mathbf{k}) + [\epsilon_m(\omega) - 1] \int S(\mathbf{k} - \mathbf{k}')\tilde{\mathbf{E}}(\mathbf{k}')\, \mathrm{d}^3 k', \tag{11}$$

where

$$S(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_V \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}^3 r \tag{12}$$

is a function which depends explicitly on the shape of the boundaries. In an infinite unbounded medium, $S(\mathbf{k}) = \delta(\mathbf{k})$ and the simple proportionality of the form $\tilde{\mathbf{D}}(\mathbf{k}) = \epsilon_m(\omega)\tilde{\mathbf{E}}(\mathbf{k})$ results. In this case, equation (9) can be solved algebraically. Applying the inverse Fourier transform to the solution, we would obtain the real-space solution in quadratures, namely,

$$\mathbf{E}(\mathbf{r}) = \frac{4\pi \mathrm{i}\omega}{c^2} \int \frac{\tilde{\mathbf{J}}_e(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})}{k^2 - (\omega/c)^2 \epsilon_m(\omega)} \frac{\mathrm{d}^3 k}{(2\pi)^3}. \tag{13}$$

This equation is mathematically similar to the transform (4) which appeared in the discussion of the toy problem of section 4.1. It can be seen that the real-space solution in an infinite absorbing medium is, necessarily, a decaying wave because the poles of the integrand in (13) are complex. These poles are obtained as the solutions to the dispersion equation $k^2 = (\omega/c)^2 \epsilon_m(\omega)$. The individual non-decaying modes $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$ in the transform (13) cannot be used to compute any physical quantity of interest. Certainly, these modes cannot be used to compute any quantity which is related to absorption, as was done in [9].

Of course, the Fourier expansion technique is superfluous in the case of an unbounded medium—one could have obtained the same result (a plane wave with the complex wave number $(\omega/c)\sqrt{\epsilon_m}$) immediately by considering the Maxwell's equations in real space. Similarly, in the case of a spatially non-uniform periodic infinite medium, the solution is a Bloch wave with a complex wavevector which must be computed by solving a suitable eigenproblem, as is well known in the theory of photonic crystals [11].

However, in the case of a finite sample of characteristic size L, the boundary-value problem is not reducible to quadratures. While the equality $S(\mathbf{k}) = \delta(\mathbf{k})$ holds, albeit approximately, for $|\mathbf{k}| \gg 1/L$, the delta-function behavior is lost for $|\mathbf{k}| \lesssim 1/L$. In order to solve the boundary-value problem correctly, we need to capture the spatial distribution of all fields in the sample on spatial scales of the order of L, and for that we need to know the spatial Fourier harmonics of the fields with $|\mathbf{k}| \lesssim 1/L$. Hence, we must use equation (11) with the correct kernel $S(\mathbf{k})$. In this case, (9) becomes an integral equation which cannot be solved algebraically. It can be seen that the Maxwell's equations for a finite object are differential in the real space but integral in the **k**-space. The equations become algebraic in the **k**-space only in an infinite unbounded medium.

If we now consider a spatially nonlocal response of the medium $\epsilon_m(\omega, \mathbf{k})$, the results would be qualitatively similar. It is apparent that the Maxwell's equations can be solved by spatial Fourier transform only in the case of infinite unbounded media. In samples of characteristic size L, a plane wave can propagate with the wavevector **k** such that $|\mathbf{k}| \gg 1/L$ under the condition that **k** satisfies the dispersion relation $k^2 = (\omega/c)^2 \epsilon_m(\omega, \mathbf{k})$. In any absorbing medium, only complex wavevectors can satisfy this equation. Therefore, the quantity $\epsilon(\omega, \mathbf{k})$ is physically meaningful only when ω and **k** are on the dispersion curve.

The conclusion that can be drawn is the following. It is not incorrect to consider a plane wave propagating in a sample of characteristic size L, provided that its wavevector satisfies $|\mathbf{k}| \gg 1/L$. Nor is it incorrect to compute various physical quantities for this wave. However, the vector **k** must satisfy the dispersion equation and is not arbitrary. It is, in fact, incorrect to choose an arbitrary purely real vector **k** (as is done in the current-driven model) and to use the plane-wave mode $\exp(i\mathbf{k} \cdot \mathbf{r})$ thus obtained to compute any observable quantity. This is especially evident for the quantities which are quadratic in the fields and, therefore, do not satisfy the superposition principle. An example of a logically flawed calculation which is based on the current-driven model (from [9]) is given in section 5.

5. Current-driven model and the heating rate

In [9], Silveirinha applies the current-driven model to compute the Poynting vector and the rate at which the medium is heated by electromagnetic radiation (the heating rate) in a periodic composite. Silveirinha claims that his derivations are completely general and first principle and uses the obtained results to criticize the earlier papers [12, 13]. In particular, Silveirinha writes that the conclusions of [12, 13] 'are founded on fundamental misconceptions and mistakes'. These claims have been contested recently on rather general grounds [14]. However [14], did not consider the technical details of Silveirinha's derivations. This will be done in this section.

All derivations of [9] which lead to results of any practical significance are carried out for Bloch waves with purely real wavevectors **k**. This assumption contradicts the well-known fact that Bloch waves in media with some amount of absorption are, necessarily, decaying and that the rate of this decay is mathematically related to the imaginary part of **k**. Thus, if **k** is taken to be purely real, the medium is, by definition, non-absorbing. Calculation of the heating rate in such a medium is meaningless: under the condition $\mathrm{Im}\,\mathbf{k} = 0$, any reasonable calculation must yield zero. Silveirinha, however, suggests that a real-valued Bloch wavevector **k** is not incompatible with losses. He argues that one can use the current-driven model to force **k** to be real, even in a lossy medium. The deficiencies of the current-driven model have been discussed above. In what follows, I will show that, if one abandons the current-driven model assumed by Silveirinha in favor of the conventional excitation scheme in which the wavevector **k** satisfies the proper dispersion equation, then Silveirinha's formulas for the heating rate yield the very result he wanted to disprove.

First, consider the method Silveirinha uses to compute the heating rate. According to equation (60) of [9], the heating rate q for a plane wave with the wavevector **k** propagating in the medium is given by the following formula:

$$q = \frac{\omega}{8\pi} \mathrm{Im}\left[\mathbf{E}^* \cdot \hat{\varepsilon}\mathbf{E} + \left(\frac{c}{\omega}\right)^2 \mathbf{E}^* \cdot \mathbf{k} \times (\hat{\mu}^{-1} - \hat{I})\mathbf{k} \times \mathbf{E}\right]. \quad (14)$$

Here I have re-written equation (60) of [9] in Gaussian units and omitted all subscripts. Tensors (dyadics) are denoted by a hat and $\hat{I}$ is the identity operator. The quantities $\hat{\varepsilon}$ and $\hat{\mu}$ are the tensors of effective permittivity and permeability of the medium. All fields are assumed to be monochromatic and the common exponential factor $\exp(-i\omega t)$ is suppressed. Equation (14) is rather general and I believe that it is correct in the limit in which the medium can be viewed as electromagnetically homogeneous.

Silveirinha evaluates (14) as follows. He uses the vector identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ to re-write the second term in the square brackets as

$$\left(\frac{c}{\omega}\right)^2 (\mathbf{E}^* \times \mathbf{k}) \cdot [(\hat{\mu}^{-1} - \hat{I})\mathbf{k} \times \mathbf{E}]. \quad (15)$$

He then uses the Maxwell's equation $\mathbf{k} \times \mathbf{E} = (\omega/c)\mathbf{B}$. It can be seen that the factor $\mathbf{k} \times \mathbf{E}$ in the end of expression (15) can be replaced by $(\omega/c)\mathbf{B}$. However, the expression $\mathbf{E}^* \times \mathbf{k}$ in the beginning of this expression can be replaced by $-(\omega/c)\mathbf{B}^*$ only if **k** is purely real. Silveirinha makes this assumption about **k** and transforms (15) to the form

$$-\mathbf{B}^* \cdot (\hat{\mu}^{-1} - \hat{I})\mathbf{B}. \quad (16)$$

The term proportional to $\hat{I}$ is then omitted since its imaginary part evaluates to zero, the field **B** is expressed in terms of the field **H** using the constitutive relation $\mathbf{B} = \hat{\mu}\mathbf{H}$, and Silveirinha arrives at the expression

$$q = \frac{\omega}{8\pi} \mathrm{Im}[\mathbf{E}^* \cdot \hat{\varepsilon}\mathbf{E} + \mathbf{H}^* \cdot \hat{\mu}\mathbf{H}], \quad (17)$$

which is, indeed, consistent with the Landau and Lifshitz textbook formula [15]. However, the transition from (14) to (17) explicitly requires that $\mathrm{Im}\,\mathbf{k} = 0$. As was shown above, **k** here is not a mathematically independent variable but must be computed from the dispersion relation which is specific to the medium. The equality $\mathrm{Im}\,\mathbf{k} = 0$ is only possible in non-absorbing media. Clearly, (17) evaluates to zero under the assumption that was used by Silveirinha to derive it. Therefore, Silveirinha cannot claim that he has confirmed the textbook result by using a more first principle or fundamental approach than the approach used by Landau and Lifshitz [15].

On the other hand, it is possible to evaluate (14) without making any assumptions about **k**. If we assume that all currents inside the medium are induced and thus abandon the unphysical current-driven model, we can use the two Maxwell's equations

$$\mathbf{k} \times \mathbf{E} = \frac{\omega}{c}\hat{\mu}\mathbf{H}, \qquad \mathbf{k} \times \mathbf{H} = -\frac{\omega}{c}\hat{\varepsilon}\mathbf{E}. \tag{18}$$

From (18), we can also obtain

$$\mathbf{k} \times \hat{\mu}^{-1}\mathbf{k} \times \mathbf{E} = -\left(\frac{\omega}{c}\right)^2 \hat{\varepsilon}\mathbf{E}. \tag{19}$$

Substitute this expression into (14). The term proportional to $\hat{\varepsilon}$ will cancel to yield

$$q = -\frac{\omega}{8\pi}\left(\frac{c}{\omega}\right)^2 \mathrm{Im}(\mathbf{E}^* \cdot \mathbf{k} \times \mathbf{k} \times \mathbf{E}). \tag{20}$$

Next, use the identity $\mathbf{k} \times \mathbf{k} \times \mathbf{E} = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - k^2\mathbf{E}$ to obtain

$$q = \frac{\omega}{8\pi}\left(\frac{c}{\omega}\right)^2 \mathrm{Im}[|\mathbf{E}|^2 k^2 - (\mathbf{E} \cdot \mathbf{k})(\mathbf{E}^* \cdot \mathbf{k})]. \tag{21}$$

This expression is equivalent to the one derived by me earlier in [12] (equation (54) of that reference). If the medium is isotropic, it can support only transverse waves whose wave number is $k^2 = (\omega/c)^2\varepsilon\mu$ with ε and μ being scalars. Then (21) is simplified to

$$q = \frac{\omega|\mathbf{E}|^2}{8\pi} \mathrm{Im}(\varepsilon\mu). \tag{22}$$

Again, this result was derived by me in [12]. The results (21) and (22) are different from the textbook expression but follow mathematically from equation (14).

Thus, it can be seen that Silveirinha's equation (14) contains the very results he wanted to disprove. The only reason Silveirinha has obtained a formula which is different from (21) or (22) is because he has used a method to evaluate (14) which is only valid when $\mathrm{Im}\,\mathbf{k} = 0$. Silveirinha's claim that the condition $\mathrm{Im}\,\mathbf{k} = 0$ is not incompatible with losses is erroneous because it is based on an excitation model which can be justified neither from the physical nor from the mathematical points of view. My method of evaluating (14) makes no assumptions about $\mathbf{k}$. It is applicable, in particular, when $\mathrm{Im}\,\mathbf{k} = 0$. In this case, (21) and (22), as well as Silveirinha's result (17), all evaluate to zero and are, in this sense, equivalent. But, unlike Silveirinha's result, formulas (21) and (22) can be used in the physically interesting case of a complex wavevector $\mathbf{k}$ and nonzero absorption.

6. Summary

In this paper, I have shown that the current-driven model of [4, 9] is not a useful mathematical tool. Its application is in some instances problematic and in others misleading or incorrect. In particular, I have shown that Silveirinha's criticism of [12, 13] is based on an error which stems directly from the use of the current-driven model. Of course, this, *per se*, does not prove the correctness of [12, 13]. However, if equation (14) (equation (60) of [9] authored by Silveirinha) is correct, then it follows with mathematical certainty that so are the results of [12, 13].

References

[1] Smith D R and Pendry J B 2006 *J. Opt. Soc. Am.* B **23** 391
[2] Silveirinha M G 2007 *Phys. Rev.* B **75** 115104
[3] Baena J D, Jelinek L, Marques R and Silveirinha M 2008 *Phys. Rev.* A **78** 013842
[4] Fietz C and Shvets G 2010 *Physica* B **405** 2930
[5] Pendry J B 2000 *Phys. Rev. Lett.* **85** 3966
[6] Ziolkowski R W and Kipple A D 2003 *IEEE Trans. Antennas Propag.* **51** 2626
[7] Pendry J B, Holden A J, Robbins D J and Stewart W J 1999 *IEEE Trans. Microw. Theory Tech.* **47** 2075
[8] Simovski C R 2009 *Opt. Spectrosc.* **107** 766
[9] Silveirinha M G 2009 *Phys. Rev.* B **80** 235120
[10] Silveiringa M G 2010 private communication
[11] Joannopoulos J D, Meade R D and Winn J N 1995 *Photonic Crystals: Molding the Flow of Light* (Princeton, NJ: Princeton University Press)
[12] Markel V A 2008 *Opt. Express* **16** 19152
[13] Richter F, Florian M and Henneberger K 2008 *Europhys. Lett.* **81** 67005
[14] Richter F, Henneberger K and Florian M 2010 *Phys. Rev.* B **82** 037103
[15] Landau L D and Lifshitz L P 1984 *Electrodynamics of Continuous Media* (Oxford: Pergamon)