Inverse problem in optical diffusion tomography. II. Role of boundary conditions

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We consider the inverse problem of reconstructing the absorption and diffusion coefficients of an inhomogeneous highly scattering medium probed by diffuse light. The role of boundary conditions in the derivation of Fourier–Laplace inversion formulas is considered. Boundary conditions of a general mixed type are discussed, with purely absorbing and purely reflecting boundaries obtained as limiting cases. Four different geometries are considered with boundary conditions imposed on a single plane and on two parallel planes and on a cylindrical and on a spherical surface.

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1. INTRODUCTION

There has been considerable recent interest in the inverse scattering problem for diffuse light. This paper is the second in a series devoted to this topic. In part I of the series[1] we established general conditions under which the existence and uniqueness of solutions to the linearized inverse problem are guaranteed and discussed various inversion formulas for imaging in a medium with free boundaries. Here we generalize these results to the case of a system in which boundary conditions are imposed at the measurement surface.

We begin by briefly summarizing the mathematical formalism of part I. We assume that the energy density $u(\mathbf{r}, t)$ of diffuse light in a medium with constant refractive index obeys the diffusion equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = \nabla \cdot [D(\mathbf{r}) \nabla u(\mathbf{r}, t)] - \alpha(\mathbf{r}) u(\mathbf{r}, t) + S(\mathbf{r}, t), \tag{1}$$

where $\alpha(\mathbf{r})$ and $D(\mathbf{r})$ are the position-dependent absorption and diffusion coefficients and $S(\mathbf{r}, t)$ is the source. We further assume that the source is harmonically modulated with frequency ω according to $S(\mathbf{r}, t) = S(\mathbf{r})[1 + A \exp(-i\omega t)]$, where $A < 1$, and the diffusion and absorption coefficients are represented as sums of their reference values D_0 and α_0 and relatively small fluctuations, according to $D(\mathbf{r}) = D_0 + \delta D(\mathbf{r})$ and $\alpha(\mathbf{r}) = \alpha_0 + \delta\alpha(\mathbf{r})$. Then, for the frequency component of $u(\mathbf{r}, t)$ at the modulation frequency ω, Eq. (1) takes the form

$$(\nabla^2 - k^2) u(\mathbf{r}) = \frac{1}{D_0} [\delta\alpha(\mathbf{r}) - \nabla \cdot \delta D(\mathbf{r}) \nabla] u(\mathbf{r}) - \frac{\gamma(A)}{D_0} S(\mathbf{r}), \tag{2}$$

where

$$\gamma(A) = \begin{cases} A & \text{if } \omega > 0 \\ 1 + A & \text{if } \omega = 0 \end{cases}, \tag{3}$$

and the diffuse wave number k is given by

$$k^2 = \frac{\alpha_0 - i\omega}{D_0}. \tag{4}$$

Hereafter, the explicit dependence of $u(\mathbf{r})$ on ω will be omitted; it is, however, implied by dispersion relation (4).

Equation (2) can be formally solved with use of the Green's function $G(\mathbf{r}, \mathbf{r}')$ according to

$$u(\mathbf{r}) = \gamma(A) \int G(\mathbf{r}, \mathbf{r}') S(\mathbf{r}') \mathrm{d}^3 r', \tag{5}$$

where $G(\mathbf{r}, \mathbf{r}')$ satisfies

$$\left[\nabla_{\mathbf{r}}^2 - k^2 - \frac{1}{D_0} \delta\alpha(\mathbf{r}) + \frac{1}{D_0} \nabla_{\mathbf{r}} \cdot \delta D(\mathbf{r}) \nabla_{\mathbf{r}}\right] G(\mathbf{r}, \mathbf{r}') = -\frac{1}{D_0} \delta(\mathbf{r} - \mathbf{r}'). \tag{6}$$

In addition to Eq. (6), the Green's function must satisfy boundary conditions on the surface of medium.

Differential equation (6) can be written conveniently in integral form as

$$G(\mathbf{r}_1, \mathbf{r}_2) = G_0(\mathbf{r}_1, \mathbf{r}_2) - \int G_0(\mathbf{r}_1, \mathbf{r})[\delta\alpha(\mathbf{r}) - \nabla_{\mathbf{r}} \cdot \delta D(\mathbf{r}) \nabla_{\mathbf{r}}] G(\mathbf{r}, \mathbf{r}_2) \mathrm{d}^3 r - D_0 \oint_S [G(\mathbf{r}, \mathbf{r}_2) \nabla G_0(\mathbf{r}_1, \mathbf{r}) - G_0(\mathbf{r}_1, \mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}_2)] \cdot \hat{\mathbf{n}} \mathrm{d}^2 r, \tag{7}$$

where the unperturbed Green's function $G_0(\mathbf{r}_1, \mathbf{r}_2)$ satisfies

$$(\nabla_{\mathbf{r}}^2 - k^2) G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{D_0} \delta(\mathbf{r} - \mathbf{r}'). \tag{8}$$

The second integral is evaluated over the surface that serves as the boundary of the volume in which all inhomogeneities are located, and $\hat{\mathbf{n}}$ is an outward-directed unit vector perpendicular to this surface (see Fig. 1 below for an illustration). If we choose G_0 to satisfy the same boundary conditions on S as G, then the above surface integral vanishes. Then Eq. (7) becomes

$$G(\mathbf{r}_1, \mathbf{r}_2) = G_0(\mathbf{r}_1, \mathbf{r}_2) - \int G_0(\mathbf{r}_1, \mathbf{r})[\delta\alpha(\mathbf{r}) - \nabla_{\mathbf{r}} \cdot \delta D(\mathbf{r}) \nabla_{\mathbf{r}}] G(\mathbf{r}, \mathbf{r}_2) \mathrm{d}^3 r. \tag{9}$$

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It will prove useful to introduce the notation

$$V(\mathbf{r}) \equiv \delta\alpha(\mathbf{r}) - \nabla_{\mathbf{r}} \cdot \delta D(\mathbf{r}) \nabla_{\mathbf{r}}. \tag{10}$$

With this notation Eq. (9) can be written as

$$G = G_0 - G_0 V G, \tag{11}$$

which has the form of the Dyson equation. Note that Eq. (9), which relates $\delta\alpha(\mathbf{r})$ and $\delta D(\mathbf{r})$ to $G(\mathbf{r}_1, \mathbf{r}_2)$, is nonlinear. It is useful to linearize Eq. (9), which to first order in V may be expressed as

$$G(\mathbf{r}_1, \mathbf{r}_2) = G_0(\mathbf{r}_1, \mathbf{r}_2) - \int G_0(\mathbf{r}_1, \mathbf{r}) V(\mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_2) \mathrm{d}^3 r. \tag{12}$$

In this paper we will investigate the inversion of integral equation (12). More specifically, we will show that it is possible to uniquely reconstruct $\delta\alpha$ and δD given a set of measurements obtained from multiple source–detector pairs located on the boundary of the medium. As in part I, the planar, cylindrical, and spherical geometries are considered separately.

The remainder of the paper is organized as follows. In Section 2 we discuss appropriate boundary conditions and derive the relation between the experimentally measurable intensity and $G(\mathbf{r}_1, \mathbf{r}_2)$. In Section 3 inversion formulas for a single plane and in Section 4 on two parallel planes are obtained. In Sections 5 and 6 we consider cylindrical and spherical geometries, respectively.

A few comments need to be made about the general approach adopted in this paper and cross referencing. We do not consider reconstruction of the diffusion and absorption coefficients separately, as was done in part I. Instead, we always consider the total data function generated by nonzero $\delta\alpha$ and δD. For this purpose we introduce the interaction operator V in Eq. (10), which depends on both coefficients. We also do not consider in detail the restriction to three-dimensional submanifolds of the four-dimensional manifold of arguments of the data function, changes of variables, and the corresponding Jacobians, since this was considered previously. References to sections of part I are made as I.X where X is the section number.

2. BOUNDARY CONDITIONS AND THE MEASURABLE INTENSITY

The intensity of diffuse light at the point $\mathbf{r}$ that flows in the direction $\hat{\mathbf{s}}$ is given by

$$I(\mathbf{r}, \hat{\mathbf{s}}) = \frac{1}{4\pi}(cu - 3D\hat{\mathbf{s}} \cdot \nabla u), \tag{13}$$

where c is the speed of light in the medium.[2,3] Far from boundaries, the second term in Eq. (13) is usually much smaller than the first and can be omitted. This is why in part I of this series[1] we assumed that the measured intensity is proportional to the density of diffuse photons, u, at the point of measurement. Near absorbing boundaries, however, the second term can be comparable to or much larger than the first. Moreover, the two terms are related to each other at the boundary. Indeed, a general boundary condition on a smooth surface S can be formulated as

$$(u + \ell \hat{\mathbf{n}} \cdot \nabla u)|_{\mathbf{r} \in S} = 0. \tag{14}$$

Here ℓ is the extrapolation distance.[4]

In the limit $\ell = 0$ we obtain purely absorbing boundaries and in the limit $\ell \to \infty$ purely reflecting boundaries. We can use Eq. (14) together with Eq. (13) to obtain the intensity I_S that is measured by detectors located on the boundary:

$$I_S(\mathbf{r}) = I_S(\mathbf{r}, \hat{\mathbf{s}} = \hat{\mathbf{n}}) = \frac{c}{4\pi}\left(1 + \frac{\ell^*}{\ell}\right) u(\mathbf{r}), \tag{15}$$

where $\ell^* \equiv 3D/c$, and we have assumed that the detectors measure the energy that flows in the direction normal to the surface, i.e., in the direction of $\hat{\mathbf{n}}$. It should be noted that in the limit $\ell \to 0$, the quantity on the right-hand side of Eq. (15) stays finite and approaches a well-defined limit. Note also that we assume that all inhomogeneities are confined inside the medium and near the boundary where the measurements are made $D = D_0$. Therefore we can use $\ell^* = 3D_0/c$ in Eq. (15).

We now consider the description of diffuse sources. Similar to detectors that measure not only the local photon density but also its gradient, the sources must be characterized not only by a point on the surface but also by a direction. In a typical experiment, photons are injected by an optical fiber that is oriented perpendicular to the boundary of the medium. Propagation of light inside a fiber is not diffuse. Instead, the fiber acts effectively as a narrow collimated beam that is described by a source term of the form (see illustration of the geometry in Fig. 1):

$$S(\mathbf{r}; \mathbf{r}_1) = \gamma(A) S_0 \delta(\boldsymbol{\rho}) f(|z - z_1|). \tag{16}$$

Here S_0 is the total source power, $\mathbf{r}_1$ is the point on the surface at which photons are injected, and $\boldsymbol{\rho}$ is the component of $\mathbf{r}$ perpendicular to $\hat{\mathbf{n}}(\mathbf{r}_1)$. Note that quantities $\boldsymbol{\rho}$ and z in Eq. (16) are defined in the local coordinate system (see Fig. 1) that is completely defined by the point $\mathbf{r}_1$ on the surface and the direction of the vector $\hat{\mathbf{n}}$ at this point. The function $f(x)$ is short ranged (typically exponential), with the following properties:

$$\int_0^\infty f(x)\mathrm{d}x = 1, \qquad \int_0^\infty x f(x)\mathrm{d}x = \ell^*. \tag{17}$$

In a strongly scattering medium the transport mean free path ℓ^* is usually small compared with characteristic sizes of the system, a requirement for the diffusion approximation to be valid. We will use this fact and the properties of the source function [Eqs. (16) and (17)] to show that

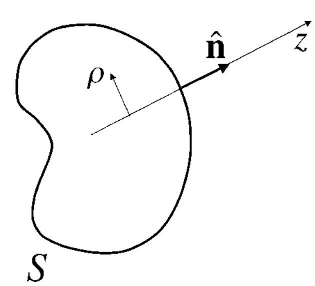


Fig. 1. Schematic illustration of the boundary conditions imposed on the surface S and the local coordinate system used in Section 2.

$$\int G(\mathbf{r}_2, \mathbf{r}) S(\mathbf{r}; \mathbf{r}_1) \mathrm{d}^3 r = \gamma(A) S_0 \left(1 + \frac{\ell^*}{\ell}\right) G(\mathbf{r}_2, \mathbf{r}_1). \tag{18}$$

To prove Eq. (18), we expand the Green's function in a Taylor series near the point $\mathbf{r} = \mathbf{r}_1$. On the line $\boldsymbol{\rho} = 0$ along which integration in Eq. (18) takes place, this expansion has the form

$$G(\mathbf{r}_2, \mathbf{r}) = G(\mathbf{r}_2, \mathbf{r}_1) - |z - z_1| \hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}_2, \mathbf{r})|_{\mathbf{r}=\mathbf{r}_1} + \dots . \tag{19}$$

From boundary condition (14) (which the Green's function must satisfy with respect to its both arguments), it follows that $\hat{\mathbf{n}} \cdot \nabla_{\mathbf{r}} G(\mathbf{r}_2, \mathbf{r})|_{\mathbf{r}=\mathbf{r}_1} = -(1/\ell) G(\mathbf{r}_2, \mathbf{r}_1)$. Substituting this into Eq. (19), we obtain

$$G(\mathbf{r}_2, \mathbf{r}) = \left(1 + \frac{|z - z_1|}{\ell}\right) G(\mathbf{r}_2, \mathbf{r}_1). \tag{20}$$

Integration of this formula over z and taking account of Eq. (17) leads directly to Eq. (18).

The above results allow us to determine the value of the measured intensity $I_S(\mathbf{r}_2)$ that is produced by a source at the point $\mathbf{r}_1$, where both $\mathbf{r}_1$ and $\mathbf{r}_2$ are on the surface:

$$I_S(\mathbf{r}_2) = \frac{c \gamma(A) S_0}{4\pi} \left(1 + \frac{\ell^*}{\ell}\right)^2 G(\mathbf{r}_1, \mathbf{r}_2). \tag{21}$$

Omitting the constant factor $c\gamma(A)S_0/4\pi$, we can define the data function as the difference in the measured intensity with and without the presence of inhomogeneities:

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = \left(1 + \frac{\ell^*}{\ell}\right)^2 [G_0(\mathbf{r}_1, \mathbf{r}_2) - G(\mathbf{r}_1, \mathbf{r}_2)]. \tag{22}$$

Finally, using Eq. (12) we arrive at

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = \left(1 + \frac{\ell^*}{\ell}\right)^2 \int G_0(\mathbf{r}_1, \mathbf{r}) V(\mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_2) \mathrm{d}^3 r, \tag{23}$$

which is the main equation that will be inverted in this paper. The data function $\phi(\mathbf{r}_1, \mathbf{r}_2)$ is directly related to the measurable quantity I_S. It is symmetric with respect to the interchange of sources and detectors.

3. HALF-SPACE GEOMETRY

We now consider boundary conditions of the kind that Eq. (14) imposed on the plane $z = 0$. Without loss of generality, we assume that all inhomogeneities are located in the right half-space ($z > 0$). In this geometry the unperturbed Green's function can be written as

$$G_0(\mathbf{r}, \mathbf{r}') = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g(q; z, z') \exp[i\mathbf{q} \cdot (\boldsymbol{\rho}' - \boldsymbol{\rho})]. \tag{24}$$

Here $\mathbf{q}$ is a two-dimensional vector parallel to the plane $z = 0$, and we have used the notation $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_z$ being a unit vector in the z direction. The function $g(q; z, z')$ must satisfy the one-dimensional equation

$$\left[\frac{\partial^2}{\partial z^2} - Q^2(q)\right] g(q; z, z') = -\frac{\delta(z - z')}{D_0}, \tag{25}$$

where

$$Q(q) \equiv (q^2 + k^2)^{1/2}. \tag{26}$$

As follows from Eq. (25), the function g is a linear combination of exponentials $\exp(\pm Qz)$ and must, in addition, satisfy the following conditions:

$$\begin{gathered} g(q; 0, z') - \ell g'(q; 0, z') = 0, \\ g(q; z' + \epsilon, z') - g(q; z' - \epsilon, z') = 0, \\ g'(q; z' + \epsilon, z') - g'(q; z' - \epsilon, z') = -1/D_0, \\ g(q; \infty, z') < \infty, \end{gathered} \tag{27}$$

where the prime denotes the first derivative with respect to z and ϵ is a positive infinitesimal constant.

The solution for g can be readily obtained:

$$g(q; z, z') = \frac{1}{2Q(q)D_0} \left\{ \frac{Q(q)\ell - 1}{Q(q)\ell + 1} \exp[-Q(q)|z + z'|] + \exp[-Q(q)|z - z'|] \right\}. \tag{28}$$

In the limiting cases $\ell = 0, \infty$, the Green's function G_0 is given by a superposition of two free-space Green's functions following the method of images. In the general case $0 < \ell < \infty$, the Green's function G_0 cannot be constructed by the method of images. However, this method still applies to its spatial Fourier components, $g(q; z, z')$, with the "reflected charge" given by $[Q(q)\ell - 1]/[Q(q)\ell + 1]$.

The definition of data function (22) [or (23)] contains only Green's functions with $\mathbf{r}_1$ or $\mathbf{r}_2$ on the boundary surface. In this case we can simplify Eq. (28) by setting z' or z to zero. Then the expression for g becomes

$$g(q; z, 0) = g(q; 0, z) \equiv \frac{\ell}{D_0} \tilde{g}(q; z) \tag{29}$$

with

$$\tilde{g}(q; z) = \frac{\exp[-Q(q)|z|]}{Q(q)\ell + 1}. \tag{30}$$

We now proceed with construction of the data function $\phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, where the two-dimensional vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ denote the locations of the sources and detectors, respectively, on the measurement surface $z = 0$. We use the definition (22) of the data function where the integration over $\mathrm{d}^3 r$ is taken over the region $z > 0$ and obtain

$$\begin{aligned} \phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \int \mathrm{d}^3 r \frac{\mathrm{d}^2 q_1 \mathrm{d}^2 q_2}{(2\pi)^4} \tilde{g}(q_1; z) \\ \times \exp[i\mathbf{q}_1 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_1)] V(\mathbf{r}) \tilde{g}(q_2; z) \\ \times \exp[i\mathbf{q}_2 \cdot (\boldsymbol{\rho}_2 - \boldsymbol{\rho})]. \end{aligned} \tag{31}$$

Next we Fourier transform the data function with respect to the two-dimensional variables $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ according to

$$\phi(\mathbf{q}_1, \mathbf{q}_2) = \int \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2 \phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \times \exp[i(\mathbf{q}_1 \cdot \boldsymbol{\rho}_1 + \mathbf{q}_2 \cdot \boldsymbol{\rho}_2)] \tag{32}$$

and obtain from Eqs. (32) and (31)

$$\phi(\mathbf{q}_1, \mathbf{q}_2) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \int \mathrm{d}^3 r \tilde{g}(q_1; z) \times \exp(i\mathbf{q}_1 \cdot \boldsymbol{\rho}) V(\mathbf{r}) \tilde{g}(q_2; z) \exp(i\mathbf{q}_2 \cdot \boldsymbol{\rho}). \tag{33}$$

Now we use the definition of $V(\mathbf{r})$ to rewrite integral equation (33) in terms of the unknown functions $\delta\alpha(\mathbf{r})$ and $\delta D(\mathbf{r})$. The action of the operator $\nabla_{\mathbf{r}} \cdot \delta D \nabla_{\mathbf{r}}$ can be easily evaluated by observing that $\nabla_{\mathbf{r}}$ is anti-Hermitian ($\nabla_{\mathbf{r}}^\dagger = -\nabla_{\mathbf{r}}$, where † denotes Hermitian conjugation) and acts to the left. Thus we have

$$\phi(\mathbf{q}_1, \mathbf{q}_2) = \int \mathrm{d}^3 r [\kappa_A(q_1, q_2; z)\delta\alpha(\mathbf{r}) + \kappa_D(q_1, q_2; z)\delta D(\mathbf{r})] \exp[i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \boldsymbol{\rho}], \tag{34}$$

where

$$\kappa_A(q_1, q_2; z) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \frac{\exp\{-[Q(q_1) + Q(q_2)]z\}}{[Q(q_1)\ell + 1][Q(q_2)\ell + 1]}, \tag{35}$$

$$\kappa_D(q_1, q_2; z) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \times \frac{[Q(q_1)Q(q_2) - \mathbf{q}_1 \cdot \mathbf{q}_2] \exp\{-[Q(q_1) + Q(q_2)]z\}}{[Q(q_1)\ell + 1][Q(q_2)\ell + 1]}, \tag{36}$$

and we have used the explicit expression for $\tilde{g}(q; z)$ [Eq. (30)].

We now investigate the existence and uniqueness of the solution to the inverse problem. We will prove that if such a solution exists, it must be unique. The existence of a solution is guaranteed if the scattering data are obtained in an "ideal experiment," i.e., are produced by some functions $\delta\alpha$ and δD according to Eq. (12). We will call the scattering data from such an experiment "physically admissible."[5] Now consider the inversion of integral equation (34). We introduce the modified data function

$$\psi(\mathbf{q}_1, \mathbf{q}_2) = \frac{D_0^2[Q(q_1)\ell + 1][Q(q_2)\ell + 1]}{(\ell + \ell^*)^2} \phi(\mathbf{q}_1, \mathbf{q}_2). \tag{37}$$

We see that Eq. (34) may be rewritten in the form

$$\psi(\mathbf{q}_1, \mathbf{q}_2) = \int \mathrm{d}^3 r \exp\{i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \boldsymbol{\rho} - [Q(q_1) + Q(q_2)]z\}\{\delta\alpha(\mathbf{r}) + [Q(q_1)Q(q_2) - \mathbf{q}_1 \cdot \mathbf{q}_2]\delta D(\mathbf{r})\}. \tag{38}$$

To see that integral equation (38) has no more than one solution for $\delta\alpha$ and δD, we must show that the null space of Eq. (38) is trivial. To this end we introduce the change of variables

$$\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2, \tag{39}$$

$$\eta_1 = Q(q_1), \tag{40}$$

$$\eta_2 = Q(q_2), \tag{41}$$

and consider the homogeneous form of Eq. (38), which is given by

$$\int \mathrm{d}^3 r \exp[i\mathbf{q} \cdot \boldsymbol{\rho} - (\eta_1 + \eta_2)z]\left\{\delta\alpha(\mathbf{r}) + \left[\eta_1\eta_2 - \frac{1}{2}(q^2 - \eta_1^2 - \eta_2^2 + 2k^2)\right]\delta D(\mathbf{r})\right\} = 0. \tag{42}$$

Evidently, if measurements of $\psi(\mathbf{q}_1, \mathbf{q}_2)$ are made at two distinct wave numbers k_1 and k_2, then Eq. (42) vanishes for k_1 and k_2 separately, and thus

$$\int \mathrm{d}^3 r \exp[i\mathbf{q} \cdot \boldsymbol{\rho} - (\eta_1 + \eta_2)z]\delta\alpha(\mathbf{r}) = 0, \tag{43}$$

$$\int \mathrm{d}^3 r \exp[i\mathbf{q} \cdot \boldsymbol{\rho} - (\eta_1 + \eta_2)z]\delta D(\mathbf{r}) = 0. \tag{44}$$

If $\delta\alpha$ and δD have compact support (in physical terms they have finite range), then the left-hand sides of Eqs. (43) and (44) are entire functions of the complex variables $\mathbf{q}$, η_1, and η_2. By analytic continuation, $\delta\alpha$, $\delta D \equiv 0$ or, equivalently, the null space of Eq. (38) is trivial, which proves the uniqueness of the solution to Eq. (38).

An inversion formula for integral equation (38) for physically admissible scattering data was derived in part I.[1] There it was shown that if Eq. (38) is transformed according to Eqs. (39)–(41) and thus holds for all choices of $\mathbf{q}$, η_1 and η_2, it must also hold for the specific choice $\eta_1 = \eta_2 = \eta/2$. Hence Eq. (38) becomes

$$\psi(\mathbf{q}, \eta/2, \eta/2) = \int \mathrm{d}^3 r \exp(i\mathbf{q} \cdot \boldsymbol{\rho} - \eta z) \times \left[\delta\alpha(\mathbf{r}) + \frac{1}{2}(\eta^2 - q^2 - 2k^2)\delta D(\mathbf{r})\right]. \tag{45}$$

Equation (45) has the structure of a Fourier–Laplace transformation, which leads to inversion formulas for $\delta\alpha$ and δD of the form

$$\delta\alpha(\mathbf{r}) = \frac{1}{k_1^2 - k_2^2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}\eta}{2\pi i} \exp(i\mathbf{q} \cdot \boldsymbol{\rho} + \eta z) \times \{k_1^2 \psi_{k_2}(\mathbf{q}, \eta/2, \eta/2) - k_2^2 \psi_{k_1}(\mathbf{q}, \eta/2, \eta/2) + (\eta^2/2 - q^2/2)[\psi_{k_1}(\mathbf{q}, \eta/2, \eta/2) - \psi_{k_2}(\mathbf{q}, \eta/2, \eta/2)]\}, \tag{46}$$

$$\delta D(\mathbf{r}) = \frac{1}{k_1^2 - k_2^2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int \frac{\mathrm{d}\eta}{2\pi i} \exp(i\mathbf{q} \cdot \boldsymbol{\rho} + \eta z) \times [\psi_{k_2}(\mathbf{q}, \eta/2, \eta/2) - \psi_{k_1}(\mathbf{q}, \eta/2, \eta/2)], \tag{47}$$

where the dependence of $\psi(\mathbf{q}_1, \mathbf{q}_2)$ on k_1 and k_2 has been made explicit. Since Eqs. (46) and (47) constitute the *unique* solution to Eq. (38) for the specific choice $\eta_1 = \eta_2 = \eta$, it follows that *no more than one* solution will satisfy Eq. (38) for arbitrary η_1 and η_2. Thus the existence and uniqueness of the solution to the inverse problem in the planar geometry is established.

4. SLAB GEOMETRY

We now consider the case when boundary conditions are imposed on two parallel planes $z = 0$ and $z = L$. The medium to be reconstructed is located between the planes in the region $0 < z < L$. Derivation of the unperturbed Green's function is similar to the one outlined in Section 3, except that the last equation in Eq. (27) (the boundary condition at infinity) is replaced by

$$g(q; L, z') + \ell g'(q; L, z') = 0. \tag{48}$$

This leads to the following expression for g:

$$g(q; z, z') = \frac{[1 + (Q\ell)^2]\cosh[Q(L - |z - z'|)] - [1 - (Q\ell)^2]\cosh[Q(L - |z + z'|)] + 2Q\ell \sinh[Q(L - |z - z'|)]}{2D_0 Q[\sinh(QL) + 2Q\ell \cosh(QL) + (Q\ell)^2 \sinh(QL)]}. \tag{49}$$

Using the fact that the Green's functions that enter the definition of the data function simplify on the measurement surface allows us to write

$$g(q; z, z')|_{z'=0,L} = g(q; z', z)|_{z'=0,L} \equiv \frac{\ell}{D_0}\tilde{g}(q; z, z'). \tag{50}$$

with

$$\tilde{g}(q; z, z') = \frac{\sinh[Q(L - |z - z'|)] + Q\ell \cosh[Q(L - |z - z'|)]}{\sinh(QL) + 2Q\ell \cosh(QL) + (Q\ell)^2 \sinh(QL)}. \tag{51}$$

The construction of the data function $\phi(\boldsymbol{\rho}_1, z_s; \boldsymbol{\rho}_2, z_d)$ in the slab geometry is analogous to the case of the half-space geometry, except that it now depends on two additional parameters, z_s and z_d, which are the z coordinates of sources and detectors, respectively. In particular, a source–detector pair can be located either on the same plane or on different planes. The expression for the Fourier-transformed data function is very similar to Eq. (33):

$$\phi(\mathbf{q}_1, z_s, \mathbf{q}_2, z_d) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \int \mathrm{d}^3 r \tilde{g}(q_1; z_s, z) \times \exp(i\mathbf{q}_1 \cdot \boldsymbol{\rho}) V(\mathbf{r}) \tilde{g}(q_2; z, z_d) \times \exp(i\mathbf{q}_2 \cdot \boldsymbol{\rho}). \tag{52}$$

Next we substitute the explicit expression for the operator V into Eq. (52). Similarly to the case of half-space geometry, we can act by the operator $\nabla_{\mathbf{r}}$ to the left (which results in a change of sign of the corresponding term since $\nabla_{\mathbf{r}}^{\dagger} = -\nabla_{\mathbf{r}}$). This yields the following integral equation:

$$\phi(\mathbf{q}_1, z_s; \mathbf{q}_2, z_d) = \int \mathrm{d}^3 r [\kappa_A(q_1, q_2, z_s, z_d; z)\delta\alpha(\mathbf{r}) + \kappa_D(q_1, q_2, z_s, z_d; z)\delta D(\mathbf{r})] \times \exp[i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \boldsymbol{\rho}], \tag{53}$$

where

$$\kappa_A(q_1, q_2, z_s, z_d; z) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \tilde{g}(q_1; z_s, z)\tilde{g}(q_2; z, z_d), \tag{54}$$

$$\kappa_D(q_1, q_2, z_s, z_d; z) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \left[\frac{\partial \tilde{g}(q_1; z_s, z)}{\partial z}\frac{\partial \tilde{g}(q_2; z, z_d)}{\partial z} - \mathbf{q}_1 \cdot \mathbf{q}_2 \tilde{g}(q_1; z_s, z)\tilde{g}(q_2; z, z_d)\right]. \tag{55}$$

Let us adduce the limiting expressions for the kernels κ_A and κ_D. For purely absorbing boundaries, the result is

$$\kappa_A(q_1, q_2, z_s, z_d; z) = \left(\frac{\ell^*}{D_0}\right)^2 \frac{\sinh[Q(q_1)(L - |z - z_s|)]\sinh[Q(q_2)(L - |z_d - z|)]}{\sinh[Q(q_1)L]\sinh[Q(q_2)L]}, \tag{56}$$

$$\kappa_D(q_1, q_2, z_s, z_d; z) = \left(\frac{\ell^*}{D_0}\right)^2 \left\{\frac{Q(q_1)Q(q_2)\Delta(z_s, z_d)\cosh[Q(q_1)(L - |z - z_s|)]\cosh[Q(q_2)(L - |z_d - z|)]}{\sinh[Q(q_1)L]\sinh[Q(q_2)L]} - \frac{\mathbf{q}_1 \cdot \mathbf{q}_2 \sinh[Q(q_1)(L - |z - z_s|)]\sinh[Q(q_2)(L - |z_d - z|)]}{\sinh[Q(q_1)L]\sinh[Q(q_2)L]}\right\}. \tag{57}$$

In the opposite limit of purely reflecting boundaries, we obtain

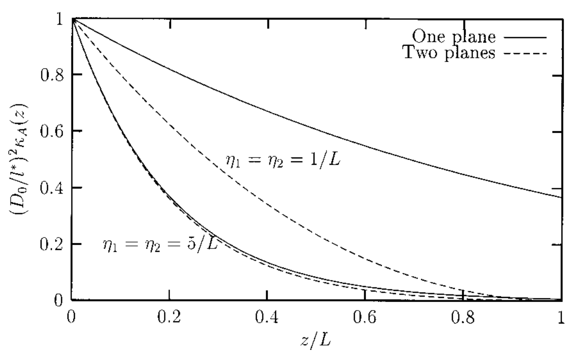
$$\kappa_A(q_1, q_2, z_s, z_d; z) = \frac{\cosh[Q(q_1)(L - |z - z_s|)]\cosh[Q(q_2)(L - |z_d - z|)]}{D_0^2 Q(q_1)Q(q_2)\sinh[Q(q_1)L]\sinh[Q(q_2)L]}, \tag{58}$$

$$\kappa_D(q_1, q_2, z_s, z_d; z) = \frac{1}{D_0^2}\left\{\frac{\Delta(z_s, z_d)\sinh[Q(q_1)(L - |z - z_s|)]\sinh[Q(q_2)(L - |z_d - z|)]}{\sinh[Q(q_1)L]\sinh[Q(q_2)L]} - \frac{\mathbf{q}_1 \cdot \mathbf{q}_2 \cosh[Q(q_1)(L - |z - z_s|)]\cosh[Q(q_2)(L - |z_d - z|)]}{Q(q_1)Q(q_2)\sinh[Q(q_1)L]\sinh[Q(q_2)L]}\right\}. \tag{59}$$

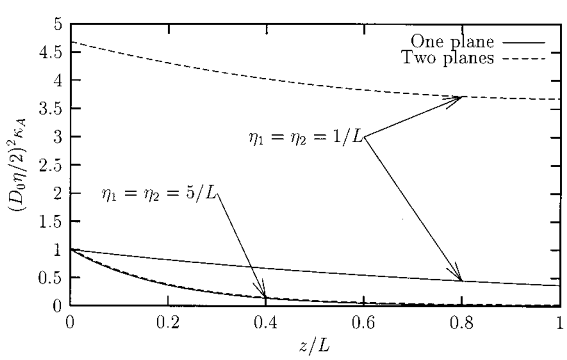
Here $\Delta(z_s, z_d) = 1$ if $z_s = z_d$ and $\Delta(z_s, z_d) = -1$ if $z_s \neq z_d$.

In principle, Eqs. (53)–(55) can be used for the purpose of reconstruction without any further modifications (e.g., by analytic singular-value decomposition[6,7]). However, the transverse part of Eq. (53) does not have the Laplace form that would guarantee the existence and uniqueness of the solution to the inverse problem. We will show below that it can be brought to the Fourier–Laplace form by taking a linear superposition of measurements with sources and detectors located on the same plane and on different planes. First, for illustrative purposes we compare the kernels κ_A for the case of one and two boundary planes. In Fig. 2 these kernels are plotted as functions of z/L assuming that $Q(q_1) = Q(q_2) = Q$ or, using new variables defined by Eqs. (39)–(41), $\eta_1 = \eta_2 = \eta$. It can be seen that the kernels for the half-space and the slab cases virtually coincide when η is large but deviate significantly when it is small or comparable to unity. We can conclude that for resolution of features on scales much smaller than L (which corresponds to large η_1, η_2) one can neglect the influence of the second boundary, assuming that both sources and detectors are located on the first boundary. However, reconstruction of the overall shape of objects on scales of the order of L requires taking the second boundary into account.

Now we show that integral equation (53) can be brought to the Fourier–Laplace form. As in part I,[1] we



(a)



(b)

Fig. 2. Kernel κ_A that appears in the transverse parts of integral equations (34) (for the half-space geometry) and (53) (for the slab geometry) as a function of z for (a) purely absorbing and (b) purely reflecting boundary conditions. The kernels are calculated for $Q(q_1) = Q(q_2) = 2\eta$. In both cases, sources and detectors are located on the plane $z = 0$.

restrict the domain of the data function to $q_1 = q_2 = q$. Then it is possible to show that

$$\kappa_A(q, q, 0, 0; z) + a(q)\kappa_A(q, q, 0, L; z) + b(q)\kappa_A(q, q, L, L; z) = c(q)\exp(-2Qz), \quad (60)$$

$$\kappa_D(q, q, 0, 0; z) + a(q)\kappa_D(q, q, 0, L; z) + b(q)\kappa_D(q, q, L, L; z) = [Q^2(q) - \mathbf{q}_1 \cdot \mathbf{q}_2]c(q) \times \exp(-2Qz), \quad (61)$$

where

$$a(q) = -2h, \qquad b(q) = h^2, \quad (62)$$

$$c(q) = \left[\frac{\ell + \ell^*}{D_0(Q\ell + 1)}\right]^2, \quad (63)$$

$$h \equiv \frac{Q\ell - 1}{Q\ell + 1}\exp(-QL). \quad (64)$$

Therefore by forming a linear combination

$$\psi(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{c(q)}[\phi(\mathbf{q}_1, 0; \mathbf{q}_2, 0) + a(q)\phi(\mathbf{q}_1, 0; \mathbf{q}_2, L) + b(q)\phi(\mathbf{q}_1, L; \mathbf{q}_2, L)]|_{q_1=q_2=q}, \quad (65)$$

we see that

$$\psi(\mathbf{q}_1, \mathbf{q}_2) = \int \mathrm{d}^3r\{\delta\alpha(\mathbf{r}) + [Q^2(q) - \mathbf{q}_1 \cdot \mathbf{q}_2]\delta D(r)\} \times \exp[i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \boldsymbol{\rho} - 2Q(q)z]. \quad (66)$$

Note that Eq. (66) has the form of integral equation (38) with $|\mathbf{q}_1| = |\mathbf{q}_2| = q$. Thus we obtain the desired Fourier–Laplace form that upon inversion establishes the existence and uniqueness of the solution to the inverse problem.

To conclude this section, we note the limiting forms for the coefficients $a(q)$, $b(q)$, and $c(q)$. For purely absorbing boundaries we obtain

$$a(q) = 2\exp[-Q(q)L], \quad (67)$$

$$b(q) = \exp[-2Q(q)L], \quad (68)$$

$$c(q) = (\ell^*/D_0)^2. \quad (69)$$

For purely reflecting boundaries, the corresponding expressions are

$$a(q) = -2\exp[-Q(q)L], \quad (70)$$

$$b(q) = \exp[-2Q(q)L], \quad (71)$$

$$c(q) = [1/D_0Q(q)]^2. \quad (72)$$

5. CYLINDRICAL GEOMETRY

We now turn our attention to the cylindrical geometry. In a cylindrical system of coordinates (z, ρ, φ) the measurement surface is specified by $\rho = R$, where R is a constant, and the medium is in the region $\rho < R$.

We start by deriving the unperturbed Green's function. It can be represented as

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}q}{2\pi} \exp[im(\varphi - \varphi')] \times \exp[iq(z - z')] g(m, q; \rho, \rho'), \quad (73)$$

where $g(m, q; \rho, \rho')$ must satisfy the one-dimensional equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} - Q^2(q)\right] g(m, q; \rho, \rho') = -\frac{\delta(\rho - \rho')}{D_0 \rho}. \quad (74)$$

The solution to Eq. (73) is given by a combination of modified Bessel and Hankel functions of the first kind, $I_m(Q\rho)$ and $K_m(Q\rho)$, and must satisfy the following conditions:

$$\begin{aligned} g(m, q; 0, \rho') &< \infty, \\ g(m, q; \rho' + \epsilon, \rho') - g(m, q; \rho' - \epsilon, \rho') &= 0, \\ g'(m, q; \rho' + \epsilon, \rho') - g'(m, q; \rho' - \epsilon, \rho') &= -1/D_0\rho', \\ g(m, q; R, \rho') + \ell g'(m, q; R, \rho') &= 0. \end{aligned} \quad (75)$$

The result is

$$g(m, q; \rho, \rho') = \frac{1}{D_0}\left[K_m(Q\rho_>)I_m(Q\rho_<) - \frac{K_m(QR) + Q\ell K_m'(QR)}{I_m(QR) + Q\ell I_m'(QR)} I_m(Q\rho)I_m(Q\rho')\right], \quad (76)$$

where $\rho_>$ and $\rho_<$ are the greater and the lesser of ρ and ρ'. On the measurement surface Eq. (76) becomes

$$g(m, q; \rho, R) = g(m, q; R, \rho) \equiv \frac{\ell}{D_0}\tilde{g}(m, q; \rho) \quad (77)$$

with

$$\tilde{g}(m, q; \rho) = \frac{1}{R}\frac{I_m(Q\rho)}{I_m(QR) + Q\ell I_m'(QR)}, \quad (78)$$

where we have used the identity

$$K_m(x)I_m'(x) - K_m'(x)I_m(x) = 1/x. \quad (79)$$

Further derivations are similar to the case of free boundaries, which is discussed in Section I.4.[1] For the sake of generality, we will use the method of reconstruction that utilizes two unit vectors (Subsection I.4.D[1]). We introduce two different and mathematically independent polar angles, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, and two corresponding unit vectors, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, that are perpendicular to the axis of the cylinder. Next, we define the Fourier-transformed data function according to

$$\phi(m_1, q_1, \tilde{\varphi}_1; m_2, q_2, \tilde{\varphi}_2) = \int_{-\infty}^{\infty} \mathrm{d}z_1 \mathrm{d}z_2 \int_0^{2\pi} \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \phi(\varphi_1, z_1; \varphi_2, z_2) \exp\{i[q_1 z_1 + q_2 z_2 + m_1(\varphi_1 - \tilde{\varphi}_1) + m_2(\varphi_2 - \tilde{\varphi}_2)]\}, \quad (80)$$

where $\phi(\varphi_1, z_1; \varphi_2, z_2)$ is defined in Eqs. (22) and (23). Transformation (80) can be easily evaluated and yields

$$\phi(m_1, q_1, \tilde{\varphi}_1; m_2, q_2, \tilde{\varphi}_2) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \int \mathrm{d}^3 r \exp\{i[m_1(\varphi - \tilde{\varphi}_1) + q_1 z]\} \times \tilde{g}(m_1, q_1; \rho) V \exp\{i[m_2(\varphi - \tilde{\varphi}_2) + q_2 z]\} \tilde{g}(m_2, q_2; \rho). \quad (81)$$

Next we define a new data function by

$$\psi(q_1, \tilde{\varphi}_1; q_2, \tilde{\varphi}_2) = \left(\frac{D_0 R}{\ell + \ell^*}\right)^2 \sum_{m_1, m_2 = -\infty}^{\infty} \{I_{m_1}[Q(q_1)R] + Q\ell I_{m_1}'[Q(q_1)R]\}\{I_{m_2}[Q(q_2)R] + Q\ell I_{m_2}'[Q(q_2)R]\} \times \phi(m_1, q_1, \tilde{\varphi}_1; m_2, q_2, \tilde{\varphi}_2) \quad (82)$$

and use the identity

$$\sum_{m=-\infty}^{\infty} I_m(w)\exp(im\theta) = \exp(w\cos\theta) \quad (83)$$

to show that

$$\psi(q_1, \hat{\mathbf{e}}_1; q_2, \hat{\mathbf{e}}_2) = \int \mathrm{d}^3 r \exp[Q(q_1)\boldsymbol{\rho}\cdot\hat{\mathbf{e}}_1 + iq_1 z] V \times \exp[Q(q_2)\boldsymbol{\rho}\cdot\hat{\mathbf{e}}_2 + iq_2 z]. \quad (84)$$

Here $\boldsymbol{\rho} = \mathbf{r} - \hat{\mathbf{e}}_z(\hat{\mathbf{e}}_z \cdot \mathbf{r})$ is a two-dimensional vector perpendicular to the axis of the cylinder characterized by the polar angle φ. As before, we evaluate the action of the operator V to arrive at

$$\psi(q_1, \hat{\mathbf{e}}_1; q_2, \hat{\mathbf{e}}_2) = \int \mathrm{d}^3 r \{\delta\alpha(\mathbf{r}) + [Q(q_1)Q(q_2)\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 - q_1 q_2]\delta D(\mathbf{r})\} \exp\{[Q(q_1)\hat{\mathbf{e}}_1 + Q(q_2)\hat{\mathbf{e}}_2]\cdot\boldsymbol{\rho} + i(q_1 + q_2)z\}. \quad (85)$$

As shown in part I,[1] arguments similar to those presented for the planar case may be used to prove the existence of the solution to integral equation (85) for physically admissable data. The proof of uniqueness of this solution is analogous to that presented for the half-space geometry and will not be further described.

6. SPHERICAL GEOMETRY

In the spherical geometry with $\mathbf{r} = (r, \theta, \varphi)$ the data function is measured on the spherical surface $r = R$ and the medium is located inside the sphere. The unperturbed Green's function can be represented as

$$G_0(\mathbf{r}, \mathbf{r}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g(l; r, r') Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}'), \quad (86)$$

where $Y_{lm}(\hat{\mathbf{r}})$ are spherical harmonics and $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector characterized by the angular variables θ and φ. The radial function $g(l; r, r')$ satisfies the one-dimensional equation

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} - k^2\right]g(l;r,r') = -\frac{\delta(r-r')}{D_0 r^2}. \tag{87}$$

The solution to Eq. (87) can be found as a linear combination of modified spherical Bessel and Hankel functions of the first kind, $i_l(kr)$ and $k_l(kR)$, and must satisfy the following conditions:

$$g(l;0,r') < \infty,$$

$$g(l;r'+\epsilon,r') - g(l;r'-\epsilon,r') = 0,$$

$$g'(l;r'+\epsilon,r') - g'(l;r'-\epsilon,r') = -1/D_0(r')^2,$$

$$g(l;R,r') + \ell g'(l;R,r') = 0. \tag{88}$$

The resulting expression for g is

$$g(l;r,r') = \frac{2k}{\pi D_0}\left[i_l(kr_<)k_l(kr_>) - \frac{k_l(kR) + k\ell k_l'(kR)}{i_l(kR) + k\ell i_l'(kR)}i_l(kr)i_l(kr')\right], \tag{89}$$

where $r_>$ and $r_<$ are the greater and the lesser of r and r'. As in the cases of the planar and cylindrical geometries, we simplify the above expression by observing that either r or r' must be equal to R in the expression for the data function:

$$g(l;r,R) = g(l;R,r) \equiv \frac{\ell}{D_0}\tilde{g}(l;r), \tag{90}$$

$$\tilde{g}(l;r) = \frac{1}{R^2}\frac{i_l(kr)}{i_l(kR) + k\ell i_l'(kR)}, \tag{91}$$

where we have used the Wronskian

$$k_l(x)i_l'(x) - k_l'(x)i_l(x) = \frac{\pi}{2x^2}. \tag{92}$$

The data function defined on the spherical surface is a function of the angular variables θ_1, φ_1, and θ_2, φ_2 of sources and detectors, respectively. For simplicity, we will use the unit vectors $\hat{\mathbf{r}}_1 = (\theta_1, \varphi_1)$ and $\hat{\mathbf{r}}_2 = (\theta_2, \varphi_2)$. Then the data function can be written as

$$\phi(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = \left(\frac{\ell + \ell^*}{D_0}\right)^2 \sum_{l_1,l_2}\sum_{m_1,m_2}\int \mathrm{d}^3 r \tilde{g}(l_1;r)Y_{l_1m_1}(\hat{\mathbf{r}}_1) \times Y^*_{l_1m_1}(\hat{\mathbf{r}})V\tilde{g}(l_2;r)Y_{l_2m_2}(\hat{\mathbf{r}})Y^*_{l_2m_2}(\hat{\mathbf{r}}_2). \tag{93}$$

Instead of Fourier transforming the data function as in the planar and cylindrical geometries, we project it onto spherical harmonics and define

$$\phi(l_1,m_1;l_2,m_2) = \int \phi(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2)Y^*_{l_1m_1}(\hat{\mathbf{r}}_1)Y_{l_2m_2}(\hat{\mathbf{r}}_2)\mathrm{d}^2\hat{\mathbf{r}}_1\mathrm{d}^2\hat{\mathbf{r}}_2. \tag{94}$$

The equation for the projected data function $\phi(l_1,m_1;l_2,m_2)$ becomes

$$\phi(l_1,m_1;l_2,m_2) = \left(\frac{\ell+\ell^*}{D_0}\right)^2\int \mathrm{d}^3 r\tilde{g}(l_1;r) \times Y^*_{l_1m_1}(\hat{\mathbf{r}})V\tilde{g}(l_2;r)Y_{l_2m_2}(\hat{\mathbf{r}}). \tag{95}$$

Now we use the expansion of the plane wave in the form

$$\exp(\mathbf{a}\cdot\mathbf{b}) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i_l(ab)Y^*_{lm}(\hat{\mathbf{a}})Y_{lm}(\hat{\mathbf{b}}) \tag{96}$$

and properties of the function $\tilde{g}$ [Eq. (91)] to show that the transformed data function

$$\psi(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2) = (4\pi)^2R^4\sum_{l_1,l_2}\sum_{m_1,m_2}[i_{l_1}(kR) + k\ell i_{l_1}'(kR)] \times [i_{l_2}(kR) + k\ell i_{l_2}'(kR)] \times Y_{l_1m_1}(\hat{\mathbf{e}}_1)Y_{l_2m_2}(\hat{\mathbf{e}}_2)\phi(l_1,m_1;l_2,m_2) \tag{97}$$

satisfies the integral equation

$$\psi(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2) = \int \mathrm{d}^3 r\exp(k\hat{\mathbf{e}}_1\cdot\mathbf{r})V\exp(k\hat{\mathbf{e}}_2\cdot\mathbf{r}) = \int \mathrm{d}^3 r[\delta\alpha(\mathbf{r}) + k^2\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2\delta D(\mathbf{r})] \times \exp[k(\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2)\cdot\mathbf{r}], \tag{98}$$

where $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are two arbitrary unit vectors.

As shown in part I,[1] the existence of the solution to integral equation (98) for physically admissible data is readily established. The proof of uniqueness of this solution is analogous to that presented for the half-space geometry and will not be repeated.

In conclusion, we have shown that the linearized inverse problem in diffusion tomography has a unique solution with arbitrary boundary conditions imposed on planar, cylindrical, and spherical surfaces. This result was achieved by bringing the integral equations for $\delta\alpha$ and δD to the Fourier–Laplace form. In general, only one modulation frequency is sufficient for reconstruction of $\delta\alpha$ when it is known *a priori* that $\delta D = 0$ and vice versa. For simultaneous reconstruction of these two coefficients, measurements with two modulation frequencies are required. In the next paper in the series we will consider the numerical inversion of the integral equations, using the method of singular-value decomposition.

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