FOURIER-LAPLACE STRUCTURE OF
THE INVERSE SCATTERING PROBLEM FOR
THE RADIATIVE TRANSPORT EQUATION

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Abstract. We consider the inverse scattering problem for the radiative transport equation. We show that the linearized form of this problem can be formulated in terms of the inversion of a suitably defined Fourier-Laplace transform. This generalizes a previous result obtained within the diffusion approximation to the radiative transport equation.

1. Introduction

1.1. Background. The inverse problem of optical tomography (OT) is to recover the optical properties of a highly-scattering medium from boundary measurements [1]. The standard approach to this problem makes use of the diffusion approximation (DA) to the radiative transport equation (RTE). Within the DA, it is possible to formulate the linearized inverse problem in terms of the inversion of a suitably defined Fourier-Laplace transform [2, 3]. Here we describe analogous results which hold beyond the DA. In particular, it is shown that by making use of the recently derived plane-wave expansion for the Green's function of the RTE [4], a generalized Fourier-Laplace structure arises in the inverse scattering problem for the RTE. This result is expected to find applications to the development of fast image reconstruction algorithms for OT with large data sets [5, 6].

This paper is organized as follows. In the remainder of this section we recall the relevant background from transport theory and review the Fourier-Laplace structure of the inverse problem within the DA. In Section 2 we give a short exposition of the plane-wave decomposition of the Green's function for the RTE. This decomposition is used in Section 3 to elucidate the generalized Fourier-Laplace structure of the linearized inverse problem for the RTE. We note that uniqueness and stability results for the nonlinear problem are discussed in [7].

1.2. Diffusion Theory. We begin by considering the propagation of light in a three-dimensional random medium. The specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$ at the point $\mathbf{r} \in V$ in the direction $\hat{\mathbf{s}}$ is assumed to obey the stationary RTE

$$(1) \qquad \hat{\mathbf{s}} \cdot \nabla I(\mathbf{r}, \hat{\mathbf{s}}) + (\mu_a + \mu_s) I(\mathbf{r}, \hat{\mathbf{s}}) = \mu_s \int p(\hat{\mathbf{s}}, \hat{\mathbf{s}}') I(\mathbf{r}, \hat{\mathbf{s}}') d^2 s' + S(\mathbf{r}, \hat{\mathbf{s}}) .$$

Here μ_a and μ_s are the absorption and scattering coefficients in V, respectively and S is the power density of the source. The phase function $p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')$ is normalized so that $\int p(\hat{\mathbf{s}}, \hat{\mathbf{s}}') d^2 s' = 1$ for all $\hat{\mathbf{s}}$ and is assumed to depend only upon the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}'$, corresponding to scattering by spherically symmetric particles. The specific intensity also satisfies a boundary condition of the form

$$(2) \qquad\qquad I(\mathbf{r}, \hat{\mathbf{s}}) = 0 , \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} < 0 , \quad \mathbf{r} \in \partial V ,$$

where $\hat{\mathbf{n}}$ is the outward unit normal to ∂V. Thus no light enters V except due to the source. The solution to (1) is given by

$$(3) \qquad\qquad I(\mathbf{r}, \hat{\mathbf{s}}) = \int d^3 r' d^2 s' G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') S(\mathbf{r}', \hat{\mathbf{s}}') ,$$

where the Green's function $G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}')$ is the solution to (1) with $S(\mathbf{r}, \hat{\mathbf{s}}) = \delta(\mathbf{r} - \mathbf{r}') \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}')$ subject to the boundary condition (2). We note that the Green's function obeys the reciprocity relation $G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = G(\mathbf{r}', -\hat{\mathbf{s}}'; \mathbf{r}, -\hat{\mathbf{s}})$.

We suppose that the medium is inhomogeneously absorbing. It is then convenient to decompose μ_a into a constant part $\bar{\mu}_a$ and a spatially varying part $\delta\mu_a$:

$$(4) \qquad\qquad \mu_a(\mathbf{r}) = \bar{\mu}_a + \delta\mu_a(\mathbf{r}) .$$

Eq. (1) can be rewritten in the form

$$(5) \quad \hat{\mathbf{s}} \cdot \nabla I(\mathbf{r}, \hat{\mathbf{s}}) + \mu_t I(\mathbf{r}, \hat{\mathbf{s}}) - \mu_s \int p(\hat{\mathbf{s}}', \hat{\mathbf{s}}) I(\mathbf{r}, \hat{\mathbf{s}}') d^2 s' = -\delta\mu_a(\mathbf{r}) I(\mathbf{r}, \hat{\mathbf{s}}) + S(\mathbf{r}, \hat{\mathbf{s}}) ,$$

where $\mu_t = \bar{\mu}_a + \mu_s$ According to (3), the solution to (5) is given by

$$(6) \qquad\qquad I(\mathbf{r}, \hat{\mathbf{s}}) = I_i(\mathbf{r}, \hat{\mathbf{s}}) - \int d^3 r' d^2 s' G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') \delta\mu_a(\mathbf{r}') I(\mathbf{r}', \hat{\mathbf{s}}') ,$$

where G denotes the Green's function for a homogeneous medium with absorption $\bar{\mu}_a$ and I_i is the incident specific intensity, defined as the solution to (1) with $\mu_a = \bar{\mu}_a$. Eq. (6) is the analog of the Lippmann-Schwinger equation of scattering theory. It describes the "multiple scattering" of the incident specific intensity from inhomogeneities in $\delta\mu_a$. If only one scattering event is considered, then the intensity I on the right hand side of (6) can be replaced by the incident intensity I_i. This result, which we refer to as the Born approximation for the RTE, linearizes the integral equation (6) with respect to $\delta\mu_a$. If the incident field is generated by a point source at $\mathbf{r}_1$ pointing in the direction $\hat{\mathbf{s}}_1$ then, within the accuracy of the Born approximation, the change in specific intensity due to spatial fluctuations in absorption can be obtained from the relation

$$(7) \qquad \phi(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}_2, \hat{\mathbf{s}}_2) = \int d^3 r d^2 s G(\mathbf{r}_1, \hat{\mathbf{s}}_1; \mathbf{r}, \hat{\mathbf{s}}) G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}_2, \hat{\mathbf{s}}_2) \delta\mu_a(\mathbf{r}) .$$

Here ϕ is proportional to the change in intensity relative to a reference medium with absorption $\bar{\mu}_a$ and $\mathbf{r}_2, \hat{\mathbf{s}}_2$ denote the position and orientation of a point detector.

The DA is obtained, following [9], by expanding the Green's function G in angular harmonics of $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}'$. To lowest order, it can be seen that

(8) $$G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}, \hat{\mathbf{s}}') = \frac{c}{4\pi} \left(1 + \ell^* \hat{\mathbf{s}} \cdot \nabla_{\mathbf{r}}\right) \left(1 - \ell^* \hat{\mathbf{s}}' \cdot \nabla_{\mathbf{r}'}\right) G(\mathbf{r}, \mathbf{r}') ,$$

where the transport mean free path $\ell^* = 1/(\bar{\mu}_a + \mu_s')$ with $\mu_s' = (1-g)\mu_s$, g being the anisotropy of the phase function p, which is assumed to be rotationally invariant. The diffusion Green's function $G(\mathbf{r}, \mathbf{r}')$ satisfies the equation

(9) $$- D\nabla^2 G(\mathbf{r}, \mathbf{r}') + \alpha G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') ,$$

where the diffusion coefficient $D = 1/3c\ell^*$ and $\alpha = c\bar{\mu}_a$. The diffusion Green's function also satisfies the boundary condition

(10) $$G(\mathbf{r}, \mathbf{r}') + \ell \hat{\mathbf{n}} \cdot \nabla G(\mathbf{r}, \mathbf{r}') = 0 , \quad \mathbf{r}, \mathbf{r}' \in \partial V ,$$

where ℓ is the extrapolation length [8]. The DA may be used to simplify the integral equation (7). Making use of (8) and assuming that the source and detector are oriented in the inward and outward normal directions, respectively, it can be seen that (7) becomes

(11) $$\phi(\mathbf{r}_1, -\hat{\mathbf{n}}(\mathbf{r}_1); \mathbf{r}_2, \hat{\mathbf{n}}(\mathbf{r}_2)) = \frac{c}{4\pi} \left(1 + \frac{\ell^*}{\ell}\right)^2 \int d^3 r\, G(\mathbf{r}_1, \mathbf{r}) G(\mathbf{r}, \mathbf{r}_2) \delta\alpha(\mathbf{r}) ,$$

where $\delta\alpha = c\delta\mu_a$. The DA is valid when the diffusion Green's function $G(\mathbf{r}, \mathbf{r}')$ varies slowly on the scale of ℓ^*. The DA breaks down in optically thin layers; in weakly scattering or strongly absorbing media, that is with $\mu_s \ll \mu_a$; and near boundaries. One or more of these conditions are often met in biomedical applications.

For the remainder of this paper we will assume that the volume V consists of the half-space $z \geq 0$. In this geometry, it can be shown that the diffusion Green's function can be expanded into two-dimensional plane waves:

(12) $$G(\mathbf{r}, \mathbf{r}') = \int \frac{d^2 q}{(2\pi)^2} e^{i\mathbf{q} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} g(z, z'; \mathbf{q}) ,$$

where we have used the notation $\mathbf{r} = (\boldsymbol{\rho}, z)$. If either $\mathbf{r} \in \partial V$ or $\mathbf{r}' \in \partial V$ then

(13) $$\begin{aligned} g(z, 0; \mathbf{q}) &= g(0, z; \mathbf{q}) \\ &= \frac{\ell}{D} \frac{1}{Q(\mathbf{q})\ell + 1} e^{-Q(\mathbf{q})z} , \end{aligned}$$

where

(14) $$Q(\mathbf{q}) = \sqrt{q^2 + \alpha/D} .$$

Note that the above diffuse modes have the form of evanescent waves which are oscillatory in the transverse direction and decay exponentially in the positive z direction.

The inverse problem of OT is to recover $\delta\mu_a$ from boundary measurements of ϕ. Within the DA, the linearized form of this problem can be studied by reduction to Fourier-Laplace form [2, 3]. To understand this point, we consider an experiment in which the source and detector are located on the $z = 0$ plane. Next, we introduce the Fourier transform of ϕ with respect to the source and detector coordinates:

(15) $$\tilde{\phi}(\mathbf{q}_1, \mathbf{q}_2) = \int d^2 \rho_1 d^2 \rho_2 e^{i(\mathbf{q}_1 \cdot \boldsymbol{\rho}_1 + \mathbf{q}_2 \cdot \boldsymbol{\rho}_2)} \phi(\boldsymbol{\rho}_1, 0, \hat{\mathbf{z}}; \boldsymbol{\rho}_2, 0, -\hat{\mathbf{z}}) .$$

It can then be seen that

$$(16) \quad \tilde{\phi}(\mathbf{q}_1, \mathbf{q}_2) = \frac{c}{4\pi} \left(\frac{\ell + \ell^*}{D} \right)^2 \frac{1}{(Q(\mathbf{q}_1)\ell + 1)(Q(\mathbf{q}_2)\ell + 1)}$$
$$\times \int d^3 r \exp \left[i \left(\mathbf{q}_1 + \mathbf{q}_2 \right) \cdot \boldsymbol{\rho} - (Q(\mathbf{q}_1) + Q(\mathbf{q}_2)) z \right] \delta\alpha(\mathbf{r}) \ .$$

Eq. (16) has the form of a Fourier-Laplace transform relating $\delta\alpha$ to $\tilde{\phi}$. This result can be used to obtain inversion formulas for the integral equation (11) [2]. Such inversion formulas are the basis for developing fast image reconstruction algorithms in OT [9].

2. Green's functions and Rotated Reference Frames

In this section we recall some details of the construction of the Green's function for the RTE using the method of rotated reference frames [4]. We begin by considering the plane-wave modes for the RTE in a homogeneous medium which are of the form

$$(17) \qquad\qquad I_{\mathbf{k}}(\mathbf{r}, \hat{\mathbf{s}}) = A_{\mathbf{k}}(\hat{\mathbf{s}}) e^{\mathbf{k} \cdot \mathbf{r}} \ ,$$

where the amplitude A is to be determined. Evidently, the components of the wave vector $\mathbf{k}$ cannot be purely real; otherwise the modes would have exponential growth in the $\hat{\mathbf{k}}$ direction. We thus consider evanescent modes with

$$(18) \qquad\qquad \mathbf{k} = i\mathbf{q} \pm \sqrt{q^2 + 1/\lambda^2}\ \hat{\mathbf{z}} \ ,$$

where $\mathbf{q} \cdot \hat{\mathbf{z}} = 0$ and $\mathbf{k} \cdot \mathbf{k} = 1/\lambda^2$. These modes are oscillatory in the transverse direction, decay in the $\pm z$-directions and are the analogs of the diffuse modes considered in (12) and (13). By inserting (17) into the RTE (1) with $S = 0$, we find that $A_{\mathbf{k}}(\hat{\mathbf{s}})$ satisfies the equation

$$(19) \qquad\qquad (\hat{\mathbf{s}} \cdot \mathbf{k} + \mu_a + \mu_s) A_{\mathbf{k}}(\hat{\mathbf{s}}) = \mu_s \int p(\hat{\mathbf{s}}, \hat{\mathbf{s}}') A_{\mathbf{k}}(\hat{\mathbf{s}}') d^2 s' \ .$$

To solve the eigenproblem defined by (19) it will prove useful to expand $A_{\mathbf{k}}(\hat{\mathbf{s}})$ into a basis of spherical functions defined in a rotated reference frame whose z-axis coincides with the direction $\hat{\mathbf{k}}$. We denote such functions by $Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}})$ and define them via the relation

$$(20) \qquad\qquad Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}) = \sum_{m'=-l}^{l} D^l_{mm'}(\varphi, \theta, 0) Y_{lm'}(\hat{\mathbf{s}}) \ ,$$

where $Y_{lm}(\hat{\mathbf{s}})$ are the spherical harmonics defined in the laboratory frame, $D^l_{mm'}$ is the Wigner D-function and φ, θ are the polar angles of $\hat{\mathbf{k}}$ in the laboratory frame. We thus expand $A_{\mathbf{k}}$ as

$$(21) \qquad\qquad A_{\mathbf{k}}(\hat{\mathbf{s}}) = \sum_{l,m} C_{lm} Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \ ,$$

where the coefficients C_{lm} are to be determined. Note that since the phase function $p(\hat{\mathbf{s}}, \hat{\mathbf{s}}')$ is invariant under simultaneous rotation of $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}'$, it may be expanded into rotated spherical functions according to

$$(22) \qquad\qquad p(\hat{\mathbf{s}}, \hat{\mathbf{s}}') = \sum_{l,m} p_l Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}) Y^*_{lm}(\hat{\mathbf{s}}'; \hat{\mathbf{k}}) \ ,$$

where the expansion coefficients p_l are independent of $\hat{\mathbf{k}}$.

Substituting (21) into (19) and making use of the orthogonality properties of the spherical functions, we find that the coefficients C_{lm} satisfy the equation

$$(23) \qquad \sum_{l',m'} R_{l'm'}^{lm} C_{l'm'} = \lambda \sigma_l C_{lm} \ .$$

Here the matrix R is defined by

$$(24) \qquad \begin{aligned} R_{l'm'}^{lm} &= \int d^2 s \, \hat{\mathbf{s}} \cdot \hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}) Y_{l'm'}^*(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \\ &= \delta_{mm'} \left(b_{lm} \delta_{l',l-1} + b_{l+1,m} \delta_{l',l-1} \right) \ , \end{aligned}$$

where

$$(25) \qquad b_{lm} = \sqrt{(l^2 - m^2)/(4l^2 - 1)} \ ,$$

and

$$(26) \qquad \sigma_l = \mu_a + \mu_s(1 - p_l) \ .$$

Eq. (23) defines a generalized eigenproblem which can be transformed into a standard eigenproblem as follows. Define the diagonal matrix $S_{l'm'}^{lm} = \delta_{mm'} \delta_{ll'} \sqrt{\sigma_l}$. Note that $\sigma_l > 0$ since $p_l \leq 1$ and thus S is well defined. We then pre and post multiply R by S^{-1} and find that $W\psi = \lambda\psi$ where $W = S^{-1}RS^{-1}$ and $\psi = SC$. It can be shown that W is symmetric and block tridiagonal with both a discrete and continuous spectrum of eigenvalues λ_μ and a corresponding complete orthonormal set of eigenvectors ψ_μ, indexed by μ [4]. We thus see that the modes (17), which are labeled by μ, the transverse wave vector $\mathbf{q}$, and the direction of decay, are of the form

$$(27) \qquad I_{\mathbf{q}\mu}^{\pm}(\mathbf{r}, \hat{\mathbf{s}}) = \sum_{l,m} \sum_{m'} \frac{1}{\sqrt{\sigma_l}} \psi_{lm}^\mu D_{mm'}^l(\varphi, \theta, 0) Y_{lm'}(\hat{\mathbf{s}}) e^{i\mathbf{q}\cdot\boldsymbol{\rho} \mp Q_\mu(\mathbf{q})z} \ ,$$

where

$$(28) \qquad Q_\mu(\mathbf{q}) = \sqrt{q^2 + 1/\lambda_\mu^2} \ .$$

The Green's function for the RTE in the half-space geometry may be constructed as a superposition of the above modes:

$$(29) \qquad G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = \int \frac{d^2 q}{(2\pi)^2} \sum_\mu \mathcal{A}_{\mathbf{q}\mu} I_{\mathbf{q}\mu}^{\pm}(\mathbf{r}, \hat{\mathbf{s}}) I_{-\mathbf{q}\mu}^{\mp}(\mathbf{r}', -\hat{\mathbf{s}}) \ ,$$

where the upper sign is chosen if $z > z'$, the lower sign is chosen if $z < z'$ and the coefficients $\mathcal{A}_{\mathbf{q}\mu}$ are found from the boundary conditions. We note that the above expression obeys the reciprocity condition. Using this result, we see that G can be written as the plane-wave decomposition

$$(30) \qquad G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}', \hat{\mathbf{s}}') = \int \frac{d^2 q}{(2\pi)^2} \sum_{lm,l'm'} g_{l'm'}^{lm}(z, z'; \mathbf{q}) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} Y_{lm}(\hat{\mathbf{s}}) Y_{l'm'}^*(\hat{\mathbf{s}}') \ ,$$

where

$$(31) \qquad \begin{aligned} g_{l'm'}^{lm}(z, z'; \mathbf{q}) &= \frac{1}{\sqrt{\sigma_l \sigma_l'}} \sum_\mu \sum_{M,M'} \mathcal{A}_{\mathbf{q}\mu} \psi_{lm}^\mu \psi_{l'm'}^\mu \\ &\quad \times D_{mM}^l(\varphi, \theta, 0) D_{m'M'}^{l'}(\varphi, \theta, 0) e^{-Q_\mu(\mathbf{q})|z-z'|} \end{aligned}$$

$$(32) \qquad \equiv \sum_\mu \mathcal{B}_{l'm'}^{lm}(\mathbf{q}, \mu) e^{-Q_\mu(\mathbf{q})|z-z'|} \ ,$$
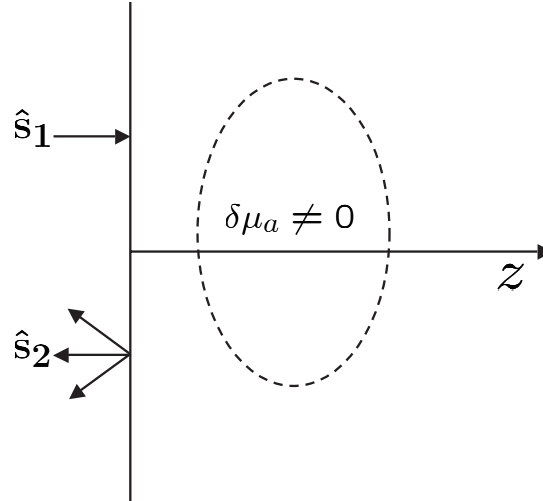
FIGURE 1. Illustrating the geometry of the experiment.

which defines $\mathcal{B}^{lm}_{l'm'}$. It is important to note that the dependence of G on the coordinates $\mathbf{r}, \mathbf{r}'$ and directions $\hat{\mathbf{s}}, \hat{\mathbf{s}}'$ is explicit and that this expansion is computable for any rotationally invariant phase function.

3. FOURIER-LAPLACE STRUCTURE

We now turn our attention to the inverse problem for the RTE. As before, we will work in the $z \geq 0$ half-space with the source and detector located on the $z = 0$ plane, as illustrated in Figure 1. The source is assumed to be pointlike and oriented in the inward normal direction. The light exiting the medium passes through a normally oriented angularly selective aperture which collects all photons with intensity

$$(33) \qquad \mathcal{I}(\mathbf{r}) = \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}>0} \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} A(\hat{\mathbf{s}}) I(\mathbf{r}, \hat{\mathbf{s}}) d^2 s \; ,$$

where A accounts for the effect of the aperture and the integration is carried out over all outgoing directions. When the aperture selects only photons traveling in the normal direction, such as occurs sufficiently far from the sample, then $A(\hat{\mathbf{s}}) = \delta(\hat{\mathbf{s}} - \hat{\mathbf{n}})$ and $\mathcal{I}(\mathbf{r}) = I(\mathbf{r}, \hat{\mathbf{n}})$. The case of complete angularly averaged data corresponds to $A(\hat{\mathbf{s}}) = 1$. If the medium is inhomgeneously absorbing, it follows from (7) and (33) that the change in intensity measured relative to a homogeneous reference medium with absorption $\bar{\mu}_a$ is given by

$$(34) \qquad \phi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}>0} \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} A(\hat{\mathbf{s}}) \phi(\boldsymbol{\rho}_1, 0, \hat{\mathbf{z}}; \boldsymbol{\rho}_2, 0, -\hat{\mathbf{s}}) d^2 s \; .$$

As before, we consider the Fourier transform of ϕ with respect to the source and detector coordinates. Upon substituting the plane wave decomposition for G given by (30) into (7) and carrying out the Fourier transform, we find, after some calculation,

that

$$(35) \quad \tilde{\phi}(\mathbf{q}_1, \mathbf{q}_2) = \sum_{\mu_1, \mu_2} M_{\mu_1\mu_2}(\mathbf{q}_1, \mathbf{q}_2) \int d^3 r \exp\left[i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \boldsymbol{\rho} \right.$$
$$\left. -(Q_{\mu_1}(\mathbf{q}_1) + Q_{\mu_2}(\mathbf{q}_2)z \right] \delta\mu_a(\mathbf{r}) \,,$$

where

$$(36) \quad M_{\mu_1\mu_2}(\mathbf{q}_1, \mathbf{q}_2) = \sum_{l_1 m_1, l_1' m_1'} \sum_{l_2 m_2} \mathcal{B}_{l_1' m_1'}^{l_1 m_1}(\mathbf{q}_1, \mu_1) \mathcal{B}_{l_2 m_2}^{l_1' m_1'}(\mathbf{q}_2, \mu_2)$$
$$\times \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}>0} \hat{\mathbf{n}} \cdot \hat{\mathbf{s}} A(\hat{\mathbf{s}}) Y_{l_2 m_2}(\hat{\mathbf{s}}) d^2 s \,.$$

Eq. (35) is the main result of this paper. It is a generalization of the Fourier-Laplace transform which holds for the DA. It can be seen that (35) reduces to (16) in the diffuse limit since only the smallest discrete eigenvalue contributes.

The inverse problem of OT consists of recovering $\delta\mu_a$ from $\tilde{\phi}$. To gain further insight into the Fourier-Laplace structure of this problem we perform the change of variables

$$(37) \quad \mathbf{q}_1 = \mathbf{q} + \mathbf{p}/2 \,, \quad \mathbf{q}_2 = \mathbf{q} - \mathbf{p}/2 \,,$$

where $\mathbf{q}$ and $\mathbf{p}$ are independent two-dimensional vectors and rewrite (35) as

$$(38) \quad \Phi(\mathbf{q}, \mathbf{p}) = \int dz K(\mathbf{q}, \mathbf{p}; z) \tilde{\delta}\mu_a(\mathbf{q}, z) \,.$$

Here $\Phi(\mathbf{q}, \mathbf{p}) = \tilde{\phi}(\mathbf{q} + \mathbf{p}/2, \mathbf{q} - \mathbf{p}/2)$, $\tilde{\delta}\mu_a(\mathbf{q}, z)$ denotes the two-dimensional Fourier transform of $\delta\mu_a$ with respect to its transverse argument and

$$(39) \quad K(\mathbf{q}, \mathbf{p}; z) = \sum_{\mu_1, \mu_2} M_{\mu_1\mu_2}(\mathbf{q} + \mathbf{p}/2, \mathbf{q} - \mathbf{p}/2)$$
$$\times \exp\left[-\left(Q_{\mu_1}(\mathbf{q} + \mathbf{p}/2) + Q_{\mu_2}(\mathbf{q} - \mathbf{p}/2) \right) z \right] \,.$$

This change of variables can be used to separately invert the transverse and longitudinal functional dependences of $\delta\mu_a$. To see this, we note that for fixed $\mathbf{q}$ (38) defines a one-dimensional integral equation for $\tilde{\delta}\mu_a(\mathbf{q}, z)$ whose pseudoinverse solution can in principle be computed numerically. We thus obtain a solution to the inverse problem in the form

$$(40) \quad \delta\mu_a(\mathbf{r}) = \int \frac{d^2 q}{(2\pi)^2} e^{-i\mathbf{q}\cdot\boldsymbol{\rho}} \int d^2 p K^+(z; \mathbf{q}, \mathbf{p}) \Phi(\mathbf{q}, \mathbf{p}) \,,$$

where K^+ denotes the pseudoinverse of K. We will consider the numerical implementation of this formula and other means of exploiting the Fourier-Laplace structure of the inverse problem elsewhere. However, we do note that the inverse problem for the RTE is ill-posed owing to the exponential decay of the evanescent modes (27) for large z. Therefore, we expect that the resolution in the z direction will degrade with depth but that sufficiently close to the surface the transverse resolution will be controlled by sampling.

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