

Topological Features of Bloch Impedance

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Abstract – The bulk-boundary correspondence (b-bc) principle states that the presence and number of *evanescent* bandgap modes at an interface between two periodic media depend on the topological invariants (Chern numbers in 2D or Zak phases in 1D) of *propagating* modes at *completely different frequencies* in all Bloch bands below that bandgap. The objective of this letter is to explain, on physical grounds, this connection between modes with completely different characteristics. We assume periodic lossless 1D structures and lattice cells with mirror symmetry; in this case the Zak phase is unambiguously defined. The letter presents a systematic study of the behavior of electromagnetic Bloch impedance, defined as the ratio of electrical and magnetic fields in a Bloch wave at the boundary of a lattice cell. The impedance-centric view confers transparent physical meaning on the bulk-boundary correspondence principle. Borrowing from the semiconductor terminology, we classify the bandgaps as *p*- and *n*-type at the Γ and X points, depending on whether the Bloch impedance has a pole (*p*) or a null (*n*) at the bottom of that gap. An interface mode exists only for *pn*-junctions per our definition. We expect these ideas to be extendable to problems in higher dimensions, with a variety of emerging applications.

Introduction. – One of the central questions in topological physics is existence of interface modes between two periodic structures at certain frequencies (energies). The bulk-boundary correspondence principle states that the presence of such modes – and, in higher dimensions, their number – depend on the topological invariants (Chern numbers in 2D or Zak phases in 1D) of all Bloch bands below the gap [1–6]. Starting from Hatsugai’s work in the early 1990s [3,4], this principle was proved in some special cases and, typically, for special boundary conditions: e.g., Harper’s equation related to the Quantum Hall effect, the Su-Schrieffer-Heeger (SSH) model, and one-dimensional (1D) structures with two layers per lattice cell [3–5].

A baffling feature of the bulk-boundary correspondence principle is that the properties of *evanescent* modes in a band gap somehow depend on the properties of *propagating* modes at *completely different frequencies*. The purpose of this letter is to demystify this connection. To this end, we, following Refs. [5, 7, 8], adopt an impedance-centric view of the topological properties of waves in periodic structures. Thus, central in our analysis is a systematic study of the behavior of electromagnetic Bloch impedance $Z(x, \omega)$ (not to be confused with the intrinsic impedance of a homogeneous material $\zeta = \sqrt{\mu/\epsilon}$). $Z(x, \omega)$ is defined

as the ratio of the electric and magnetic fields $e(x)$ and $h(x)$ of a Bloch wave at a given frequency ω . Particular attention is paid to the value $Z(0, \omega)$, $x = 0$ being the lattice cell boundary and also, for two abutting media, their interface. In the latter case, the Bloch impedances are in general different and will be denoted with $Z_{1,2}(0, \omega)$.

Throughout the letter, we assume lossless materials and lattice cells with a mirror symmetry. It is in this case that topological properties manifest themselves in their purest form; in particular, the Zak phase is unambiguously defined [1,9]. A great deal of interest in more general situations notwithstanding – in particular, higher dimensions and non-Hermitian systems ([6,10,11] and references therein), – rigorous and generalizable results for simpler models may serve as a foundation for the analysis of more complex ones.

Consider two semi-infinite periodic 1D structures with a common interface $x = 0$ (Fig. 1), and let the frequency ω be in a band gap of both media (which we assume to have overlapping band gaps). A key question is how the possible presence of an evanescent boundary mode is related to the properties of propagating waves at lower frequencies. A conceptual flow of our analysis is illustrated in Fig. 2. The starting point is to note the importance of the

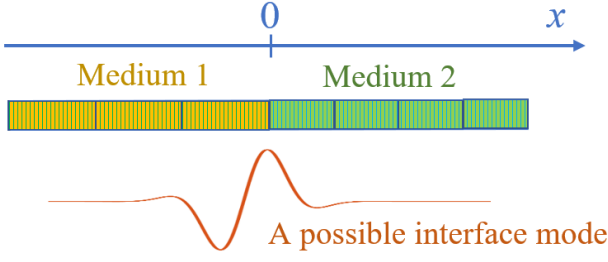


Fig. 1: Schematic illustration of an evanescent mode between two semi-infinite periodic 1D structures with a common interface.

boundary value of the Bloch wave impedance $Z(0, \omega)$ [5]. Indeed, Maxwell's boundary condition for the existence of an evanescent edge mode is $Z_1(0, \omega) + Z_2(0, \omega) = 0$.

Crucially, at the Γ and X points within any band gap, $Z(0, \omega)$ turns out to be imaginary (in lossless media), and its imaginary part $Z_I(0, \omega)$ decreases monotonically with ω in one of the two ways: either as $+\infty \rightarrow 0$ (pole-to-null) or as $0 \rightarrow -\infty$ (null-to-pole). Furthermore, poles of $Z(0, \omega)$ at the Γ and X points correspond to Bloch modes with a symmetric field $e(x)$, while nulls correspond to antisymmetric modes, where symmetry is defined as $e(x)$ being even, $e(x) = e(-x)$ for all x . At the same time, for cells with a mirror symmetry, the Zak phase of a Bloch band is related to the parity change of the modes between the Γ and X points across that band [1, 9] and, hence, to either a null-to-pole or pole-to-null transition of $Z(0, \omega)$. The bulk-boundary correspondence principle is derived by connecting all these observations.

Preliminaries (1D electromagnetic problem). – We assume normal propagation of monochromatic waves in a medium whose properties vary only in the x -direction; each of the fields $\mathbf{e}(x)$, $\mathbf{h}(x)$ depends only on x and has only one Cartesian component. The complex amplitudes $e(x)$ and $h(x)$ satisfy the ordinary differential equations,

$$e'(x) = ik\mu(x)h(x), \quad h'(x) = ik\epsilon(x)e(x), \quad (1)$$

in the Gaussian system under the $\exp(-i\omega t)$ phasor convention; $k = \omega/c$ is the wave number. The dielectric permittivity $\epsilon(x)$ and magnetic permeability $\mu(x)$ are assumed to be real, strictly positive, bounded away from zero, lattice-periodic, and possess mirror symmetry:

$$\begin{aligned} \epsilon(x+a) &= \epsilon(x), \quad \mu(x+a) = \mu(x); \\ \epsilon(a-x) &= \epsilon(x), \quad \mu(a-x) = \mu(x). \end{aligned} \quad (2)$$

These properties hold for all x , $x+a$, $a-x$ within a given medium, and a is its lattice period. Dependence of physical quantities on the frequency ω may not always be explicitly indicated. System (1) can be converted to a second-order equation for either of the fields,

$$\mathcal{L}_e e(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon(x)} \frac{d}{dx} \frac{1}{\mu(x)} \frac{de(x)}{dx} + k^2 e(x) = 0 \quad (3)$$

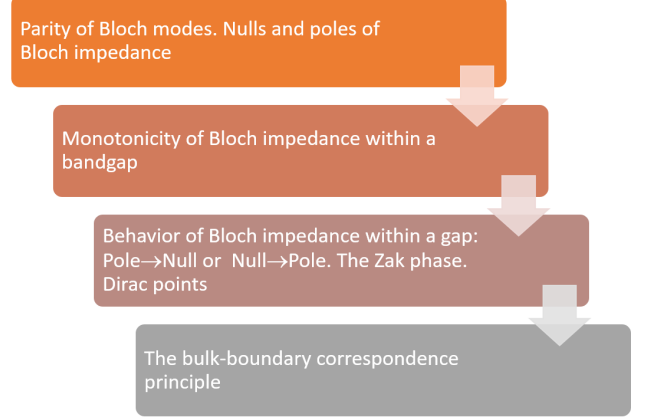


Fig. 2: Conceptual flow of analysis.

$$\mathcal{L}_h h(x) \stackrel{\text{def}}{=} \frac{1}{\mu(x)} \frac{d}{dx} \frac{1}{\epsilon(x)} \frac{dh(x)}{dx} + k^2 h(x) = 0. \quad (4)$$

The standard Bloch boundary conditions read

$$\begin{aligned} e(a) &= \lambda e(0) = \exp(iqa) e(0), \\ h(a) &= \lambda h(0) = \exp(iqa) h(0), \end{aligned} \quad (5)$$

where $\lambda = \exp(iqa)$ is the eigenvalue and q is the Bloch wave number. Both quantities are, in general, complex, but under the assumptions of this letter, q is either real or imaginary. It is standard to restrict the real part of q to the first Brillouin zone (FBZ):

$$\text{FBZ:} \quad -\pi < \text{Re}(qa) \leq \pi. \quad (6)$$

An essential role in the analysis will be played by the impedance $Z(x)$ and its inverse, the admittance $Y(x)$, of a given Bloch wave. These quantities are defined as

$$Z(x) \stackrel{\text{def}}{=} e(x)/h(x), \quad Y(x) \stackrel{\text{def}}{=} h(x)/e(x). \quad (7)$$

Especially important will be the boundary values $Z(0) = Z(a)$ and $Y(0) = Y(a)$. If either $h(x) = 0$ or $e(x) = 0$, we say that either the impedance or the admittance has a pole. Poles and nulls of the the boundary values $h(0)$ and $e(0)$ at the Γ and X points are of particular interest.

Properties of the transfer matrix. – It is convenient to merge $e(x)$ and $h(x)$ into a single vector

$$\psi(x) = \begin{bmatrix} e(x) \\ h(x) \end{bmatrix}. \quad (8)$$

Since Eqs. (1) are linear in the fields, the vector $\psi(x)$ is related to $\psi(0)$ by a linear transformation

$$\psi(x) = T(x) \psi(0), \quad (9)$$

where $T(x)$ is a 2×2 transfer matrix. The columns of $T(x)$ are the two fundamental solutions $\psi_{10}(x)$, $\psi_{01}(x)$ of (3) satisfying the boundary conditions

$$\psi_{10}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_{01}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (10)$$

For multilayered structures, a closed form of the transfer matrix can be found by matching the plane-wave solutions to (3), (4) layer-by-layer [12]. For a general variation of $\epsilon(x)$ and $\mu(x)$, various approximations can be used. The special value $T(a)$ is known as the *monodromy matrix*; its properties, summarized below, play an important role in our analysis.

1. For a single homogeneous layer of width δ , refractive index $n = \sqrt{\epsilon\mu}$ and intrinsic impedance $\zeta = \sqrt{\mu/\epsilon}$, the transfer matrix is of the form

$$T(\delta) = \begin{bmatrix} \cos(nk\delta) & i\zeta \sin(nk\delta) \\ i\zeta^{-1} \sin(nk\delta) & \cos(nk\delta) \end{bmatrix}. \quad (11)$$

2. For a general (not necessarily homogeneous) lossless cell of width a , the monodromy matrix is of the form

$$T(a) = \begin{bmatrix} \alpha & i\beta \\ i\gamma & \alpha \end{bmatrix} \quad (12)$$

with real α, β, γ .

3. The determinant of $T(a)$ is equal to unity, viz,

$$\det T(a) = \alpha^2 + \beta\gamma = 1. \quad (13)$$

4. The characteristic equation of $T(a)$ is

$$\lambda^2 - 2\alpha\lambda + 1 = 0 \quad (14)$$

with the roots

$$\lambda_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1}. \quad (15)$$

Note that $\lambda_1\lambda_2 = 1$. The real reciprocal roots for $|\alpha| > 1$ correspond to a band gap, while the complex conjugate roots for $|\alpha| < 1$ correspond to a pass band.

5. For $\alpha = 1$ or $\alpha = -1$, it is true that $qa = 0$ or $|qa| = \pi$, respectively, while the monodromy matrix acquires the Jordan form

$$T(a) = \begin{bmatrix} 1 & i\beta \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad T(a) = \begin{bmatrix} 1 & 0 \\ i\gamma & 1 \end{bmatrix}. \quad (16)$$

6. In a band gap, the impedance $Z(0)$ of an evanescent Bloch mode is imaginary, $Z(0) = iZ_I(0)$, with

$$Z_I(0) = -\text{sgn}(\alpha) \text{sgn}(\beta) \sqrt{-\beta/\gamma}, \quad (17)$$

and $\text{sgn}(\beta) = -\text{sgn}(\gamma)$. The sign of $Z_I(0)$ can vary.

Since admittance and impedance are inverses of each other, similar results hold for the admittance $Y(0)$. Proofs of the above assertions are outlined below.

1. Expression (11) for a homogeneous layer follows from writing $e(x) = c_1 \exp(iknx) + c_2 \exp(-iknx)$, finding the coefficients c_1, c_2 from the boundary conditions at $x = 0$, and then substituting $x = \delta$.

2. For a single homogeneous layer, (12) is evident from (11). If $T(a)$ of a given structure has the form (12), then adding two layers of width δ symmetrically yields $T(a + 2\delta) = T(\delta)T(a)T(\delta)$. Direct calculation shows that this product is also of the form (12); hence (12) follows by induction for a general symmetric layered cell. Smooth spatial variation of $\epsilon(x)$ and $\mu(x)$ can be viewed as a limiting case of that, whereby the layer thicknesses tend to zero.
3. Given that $T(x) = [\psi_{10}(x), \psi_{01}(x)]^T$, direct differentiation yields $[\det T(x)]' = 0$ and, since $T(0)$ is the identity matrix, $\det T(x) = 1$ [7, 8]. Alternatively, this result follows from the Abel–Liouville–Jacobi–Ostrogradskii identity for the Wronskian of a linear system [13, §8.4].
4. Eqs. (14) and (15) follow from (13) by straightforward algebra.
5. Assume that $\alpha = \pm 1$. Since $\det T(a) = \alpha^2 + \beta\gamma = 1$ according to (13), we have $\beta\gamma = 0$, which implies the Jordan form (16).
6. By definition of the transfer matrix (9), we have

$$(\alpha - \lambda)e(0) + i\beta h(0) = 0. \quad (18)$$

It follows that

$$Z(0) = e(0)/h(0) = -i\beta/(\alpha - \lambda). \quad (19)$$

An evanescently decaying mode has the Bloch eigenvalue $|\lambda| < 1$. Then, from (15) and (13),

$$\lambda = \alpha - \text{sgn}(\alpha)\sqrt{-\beta\gamma}, \quad (20)$$

where $\text{sgn}(\beta) = -\text{sgn}(\gamma)$. Substituting this into (19), we obtain

$$Z(0) = -i \text{sgn}(\alpha) \text{sgn}(\beta) \sqrt{-\beta/\gamma}, \quad (21)$$

which is equivalent to (17).

Properties of solutions. – Assuming mirror-symmetric cells *and* Bloch modes at the Γ and X points (important: both assumptions must hold), these modes are either symmetric (S), $e(-x) = e(x)$, or antisymmetric (AS), $e(-x) = -e(x)$ [7, 8, 14]. To prove that, it is convenient to define the mirror symmetry operator \mathcal{P} as

$$\mathcal{P}e(x) \stackrel{\text{def}}{=} e(-x). \quad (22)$$

It is easy to show that \mathcal{P} commutes with either of the differential operators $\mathcal{L}_e, \mathcal{L}_h$. Since the boundary conditions are also \mathcal{P} -invariant at Γ and X , at these points one can construct a pair of even and odd Bloch modes

$$e_S(x) = \frac{e(x) + e(-x)}{2}, \quad e_{AS}(x) = \frac{e(x) - e(-x)}{2}. \quad (23)$$

If the Bloch mode $e(x)$ is not identically zero, there are two possibilities:

1. $e_S(x)$ and $e_{AS}(x)$ are both nontrivial. In this case, these two modes form a complete set of solutions of (1) and the respective Bloch problem.
2. One of the modes $e_S(x)$, $e_{AS}(x)$ is identically zero. In this case, the Bloch problem has only one solution of either even or odd parity.

Note that the case of both $e_S(x)$ and $e_{AS}(x)$ being identically zero can be excluded, as this would imply $e(x) \equiv 0$. Since there are, typically, two Bloch modes propagating in the opposite directions, Case 1 above may appear to be general, and Case 2 exceptional. However, at the Γ and X points it is the other way around. Indeed, two linearly independent Bloch modes may exist – or, equivalently, the monodromy matrix of the Jordan form (16) may be non-defective – only in the exceptional case when this matrix is diagonal, and, even more specifically (since $|\alpha| = 1$ at $qa = 0, \pi$), $T(a) = \pm \mathbb{I}_2$, where \mathbb{I}_2 is the 2×2 identity matrix.

Parity of modes and Bloch impedance in band gaps.

There is a transparent relation between the parity of Bloch modes at the Γ and X points on the one hand and the Bloch impedance/admittance $Z(0)$, $Y(0)$ at the cell boundary on the other. For symmetric modes, $e(0) = 1$, $h(0) = 0$; hence $Y(0) = 0$ and $Z(0)$ has a pole. The opposite is true for antisymmetric modes.

Central in our analysis is the behavior of Bloch impedance in band gaps. As proved in [7], *the parity of Bloch modes at the Γ and X points changes across any band gap*. This implies that the Bloch impedance changes from a pole to zero, or vice versa, as the frequency increases across a band gap. Let us, following Refs. [7, 8], introduce functions $\xi(x)$, $\chi(x)$, which can be referred to as the *mathematical* impedance and admittance:

$$\begin{aligned} \xi(x) &\stackrel{\text{def}}{=} -\frac{\mu(x)e(x)}{e'(x)} = -\frac{e(x)}{ikh(x)} = k^{-1}Z_I(x), \\ \chi(x) &\stackrel{\text{def}}{=} -\frac{\epsilon(x)h(x)}{h'(x)} = -\frac{h(x)}{ike(x)} = k^{-1}Y_I(x). \end{aligned} \quad (24)$$

We immediately see that

$$k^2\xi(x)\chi(x) = Z_I(x)Y_I(x) = -Z(x)Y(x) = -1. \quad (25)$$

It is proved in [8] that the *mathematical impedance* $\xi(0)$ monotonically decreases as a function of k within any band gap. Even though this theorem is sufficient for the analysis of bulk-boundary correspondence, the following stronger result on the monotonicity of $Z_I(0) = k\xi(0)$ is instructive:

The Bloch wave impedance $Z_I(0)$ monotonically decreases as a function of frequency within any band gap.

Indeed, in a gap, the Bloch eigenvalue λ is real ($|\lambda| < 1$ for the decaying mode). Since the material parameters are assumed real and the Bloch boundary conditions are real as well, solutions to the Bloch problem can be chosen as real. Thus $\chi(0)$, $\xi(0)$, $Z_I(0)$, $Y_I(0)$ are all real. If $\xi(0)$ happens to be negative and is, due to [8], monotonic, then $Z(0) = k\xi(0)$ monotonically decreases as well.

Now suppose that $\xi(0)$ is positive. Applying the monotonicity argument to the magnetic field equation (4), we conclude that $\chi(0)$ is monotonically decreasing with frequency. But, as follows from (25), if $\xi(0)$ is positive, then $\chi(0)$ must be negative. Hence $Y_I(0) = k\chi(0)$ is also monotonically decreasing, and so does $Z_I(0) = -1/Y_I(0)$.

Nulls, poles, and the behavior of Bloch impedance in band gaps. Since, within a band gap, the Bloch wave impedance at the Γ and X points decreases monotonically with ω either from a pole to a null or vice versa, there are only two possibilities for this change: from $+\infty$ to 0 or from 0 to $-\infty$. We introduce the following definition.

Definition 1 (*p- and n-type band gaps*). *We say that a band gap is of type p if the Bloch impedance $Z(0)$ has a pole (p) at the bottom (lowest frequency) of that gap, or, equivalently, the Bloch mode at the bottom of the gap is symmetric. (Then $Z(0)$ must have a null at the top of the gap.) Similarly, we say that a band gap is of type n if the Bloch impedance $Z(0)$ has a null at the bottom of the gap, with the respective mode being antisymmetric. (Connection with semiconductor terminology is purely mnemonic and not substantive.)*

We illustrate this with a Bloch band diagram of a three-layer nonmagnetic structure with the permittivities of the layers $\epsilon_{1,2,3} = 1.00, 4.87, 1.00$ and widths $d_{1,2,3} = 0.25a, 0.50a, 0.25a$ (Fig. 3). The Γ and X points in the diagram are color-coded. The cyan dots indicate symmetric modes and the respective poles of $Z(0)$; the black dots indicate antisymmetric modes and nulls of $Z(0)$. The points across each gap have opposite color, in accordance with the theory. Per Definition 1, the gaps with black dots at the bottom are of the type *n*, while the gaps with cyan dots at the bottom are of the type *p*. Same color-coding for any two band gaps indicates the same transition of $Z(0)$ from the bottom to the top of the gap: either from $+\infty$ to 0 or, alternatively, from 0 to $-\infty$.

At the bottom of the first band gap ($qa = \pi$, $ka \approx 1.53$), the mode is antisymmetric and, consequently, $Z(0) = 0$. Therefore, in the first band gap ($1.53 \lesssim ka \lesssim 2.32$) $Z_I(0)$ must monotonically decrease from 0 to $-\infty$, as indeed is evident from Fig. 4, where $Z(0)$ is plotted as a function of ka for the same layered medium. The behavior of $Z(0)$ in the second and third gap is the same, as is symbolically indicated by the black-to-cyan dot transition across these gaps. But the fourth gap ($qa = 0$, $7.47 \lesssim ka \lesssim 8.14$) is different: namely, the cyan-to-black transition indicates that $Z(0)$ decreases monotonically from $+\infty$ to 0.

Let us note in passing that, as evidenced by Fig. 4, *the Bloch impedance does not change sign within a band*. Although this result is not used directly in our analysis, it is interesting in its own right and is proved in the Appendix.

Definition 2 (*p–n, p–p and n–n junctions*). *In our 1D setup, suppose that two periodic media share a common interface. Suppose further that a given frequency lies in some band gap of each medium. Under these assumptions,*

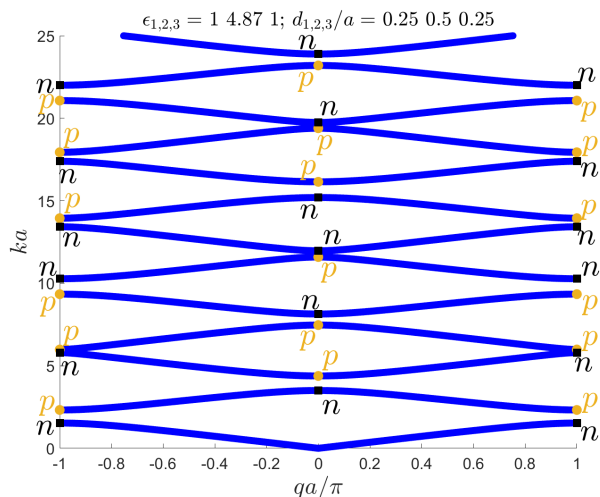


Fig. 3: Bloch bands in the positive half of the FBZ for a symmetric three-layer, non-magnetic lattice cell. Permittivities $\epsilon_{1,2,3} = 1.00, 4.87, 1.00$; widths $d_{1,2,3} = a/4, a/2, a/4$.

we say that the junction is of the type $p - n$ if the band gaps in the two media are of opposite types (per Definition 1). The $p - p$ and $n - n$ junctions are defined analogously.

This definition helps to address a key point in the analysis of bulk-boundary correspondence: whether two consecutive gaps are of the same type and hence exhibit the same behavior of $Z_I(0)$. The answer depends on whether the mode parity changes across the band between these two gaps. If the two ends of the band are of different type, then the behavior of $Z(0)$ in the gaps is the same – as, e.g., for band 2 connecting gaps 1 and 2 in Fig. 3. Conversely, if the parity of the modes across the band is the same, then the behavior of $Z(0)$ in the two consecutive gaps is opposite; an example of such behavior is band 4.

The bulk-boundary correspondence principle. –

To formalize the results of the previous section, let us introduce a binary index γ_m of the band gap number m :

$$\gamma_m = \begin{cases} 1, & Z_I(0): +\infty \rightarrow 0 \quad (p \rightarrow n) \\ 0, & Z_I(0): 0 \rightarrow -\infty \quad (n \rightarrow p) \end{cases}, \quad (26)$$

where the arrows indicate the change of impedance as k increases across the gap. Clearly, the imaginary part of impedance is positive inside the gap if $\gamma_m = 1$ and negative otherwise. Similarly, let us assign the index s_j (s for “same”) to the Bloch band j as follows:

$$s_j = \begin{cases} 1, & \text{same parity of modes at } \Gamma \text{ and } X \\ 0, & \text{different parity of modes at } \Gamma \text{ and } X \end{cases}. \quad (27)$$

There exists a simple one-to-one correspondence between s_j and the Zak phase (see Eq. 31 below). In the absence of Dirac points (accidental band crossings), we have

$$\gamma_m = \sum_{j=1}^m s_j \pmod{2}. \quad (28)$$

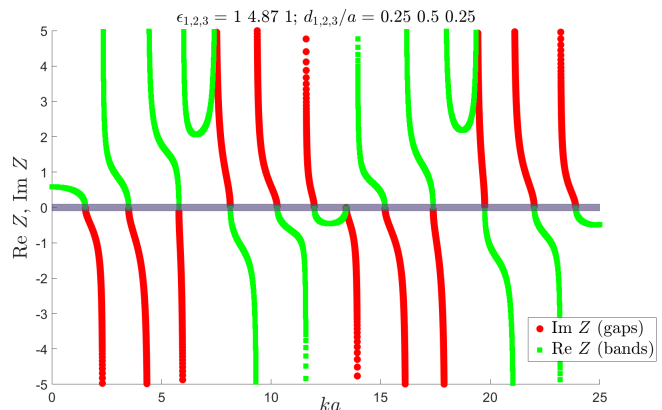


Fig. 4: The Bloch impedance $Z(0)$ (real within the bands; purely imaginary in the gaps) as a function of frequency. Same parameters as in Fig. 3.

Indeed, at $k = q = 0$, $e(x) = \text{const}$; this mode is trivially symmetric. If $s_1 = 1$, then, by definition, the mode at the bottom of gap 1 is also symmetric, and $\gamma_1 = 1$. Thus, (28) holds for $m = 1$. The rest follows by induction using similar reasoning. The case when Dirac points are present is considered below.

We are now in a position to analyze the existence of an interface mode between two semi-infinite periodic media, with the boundary impedances $Z_1(0, \omega) = iZ_{I1}(0, \omega)$ and $Z_2(0, \omega) = iZ_{I2}(0, \omega)$, where $Z_{I1,2}(0, \omega)$ are real. It follows from Maxwell’s boundary conditions that an evanescent interface mode exists if and only if

$$\Sigma(\omega) \stackrel{\text{def}}{=} Z_{I1}(0, \omega) + Z_{I2}(0, \omega) = 0. \quad (29)$$

To set the stage, consider an interface of a non-magnetic homogeneous dielectric with a semi-infinite periodic medium at a frequency within the band gap m of the latter. If $\gamma_m = 0$, then $Z_I(0, \omega)$ changes from 0 to $-\infty$; due to the continuity of that change, there must be a value ω at which $Z_I(0, \omega) = -\zeta$, where $\zeta = \sqrt{1/\epsilon}$ is the intrinsic impedance of the dielectric. At that frequency, (29) is satisfied, and an evanescent interface mode must exist. If, on the other hand, $\gamma_m = 1$, then $Z_I(0, \omega)$ changes from $+\infty$ to 0, (29) cannot hold, and there are no boundary modes.

This analysis can be generalized to an interface between two semi-infinite periodic media. Let there be a (partial of full) overlap $\Omega = [\omega_1, \omega_2]$ of the band gaps of the two media. If the γ -indexes of the respective gaps are the same, then the two impedances have the same sign in Ω and the interface condition (29) cannot hold.

Let us now consider the case where the γ indexes of the two media in their respective gaps are different. Then $Z_I(0, \omega)$ of one of these media changes from $+\infty$ to 0, whereas the other one changes from 0 to $-\infty$. It is intuitively clear that, under such conditions, there has to be a frequency $\omega_0 \in \Omega$ such that $\Sigma(\omega_0) = 0$; then an interface mode exists at $\omega = \omega_0$. To show this rigorously, note that ω_1 must be either a pole or a null of at least one of the

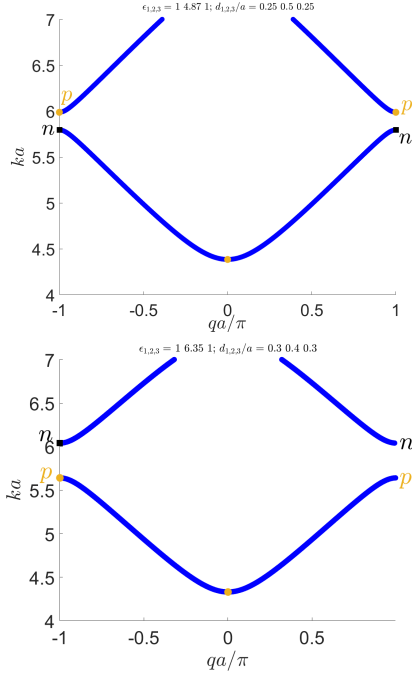


Fig. 5: Example: two three-layer media with overlapping bandgaps of different type. Top: an n -type gap; permittivities $\epsilon_{1,2,3} = 1, 4.87, 1$; layer widths $w_{1,2,3} = 0.25a, 0.5a, 0.25a$. Bottom: a p -type gap; $\epsilon_{1,2,3} = 1, 6.35, 1$; $w_{1,2,3} = 0.3a, 0.4a, 0.3a$.

impedances. Without a loss of generality, assume that ω_1 is a pole of $Z_1(0, \omega)$ (the case of a null is analogous). Then

$$\Sigma(\omega) = Z_{I1}(0, \omega) + Z_{I2}(0, \omega) \rightarrow +\infty \text{ as } \omega \rightarrow \omega_1, \quad (30)$$

where $Z_{I1}(0, \omega) > 0$ and $Z_{I2}(0, \omega) < 0$. At the other end of the overlap region, ω_2 , there are two distinct possibilities:

1. ω_2 is a null of $Z_1(0, \omega)$. Then $\Sigma(\omega_2) = Z_{I2}(0, \omega_2) < 0$. Consequently, $\Sigma(\omega)$ has opposite signs and the ends of the overlap interval and must be equal to zero at some intermediate point $\omega_0 \in \Omega$.
2. ω_2 is a pole of $Z_2(0, \omega)$. Then $\Sigma(\omega) \rightarrow -\infty$ as $\omega \rightarrow \omega_2$. Consequently, $\Sigma(\omega)$ has opposite signs at ω_1 and ω_2 , leading to the same conclusion as in Case 1.

We have therefore arrived at the following result: If two periodic media share a common interface and the given frequency lies in band gaps of both media, an interface mode exists if and only if the junction is of the $p-n$ type according to Definition 2.

Fig. 5 gives a numerical illustration of two periodic media whose bandgaps just below $ka \approx 6$ are of different type; put together, these media would form a $p-n$ junction in the overlap range of the gaps – e.g., for $ka \approx 5.9$. The material parameters are indicated in the figure caption.

Parity of Bloch modes and Zak phase. – A widely accepted parameter, closely related to the change of parity of Bloch modes across a band, is the Zak phase θ [1, 5, 9].

It is a particular case of the Berry phase, whereby the Bloch wave number is treated as an independent variable crossing the FBZ. As such, the Zak phase is commonly defined via an integral of the Berry connection, or the limit of the corresponding discrete sum of phase shifts between the consecutive eigenmodes over the FBZ [1, 9]. For symmetric lattice cells, θ can assume only two values: 0 or π , depending on whether the parity of a Bloch mode across the band changes (then $\theta = \pi$) or not (then $\theta = 0$). Thus, there is a one-to-one correspondence with the index s_j defined in (27):

$$s_j = 1 - \theta_j/\pi \quad (31)$$

With this in mind, (28) can be re-written in the form

$$s_m = (-1)^{m-1} \sum_{j=1}^m \exp(i\theta_j) \pmod{2}. \quad (32)$$

This result appears in Refs. [5, 7, 8].

The role of Dirac points. – Dirac points (accidental degeneracies or band crossings) may occur at Γ and X , but not strictly inside the FBZ. Indeed, if the latter were the case, in the vicinity of such a crossing point there would be four possible values of the Bloch wave number, which is impossible. For cells with two different constituent materials, the following explicit analytical condition for the existence of Dirac points is available [5]. If the intrinsic impedances of the two constituents are different, then Dirac points exist if and only if the optical lengths of the constituents are commensurate – that is, $(n_1 d_1)/(n_2 d_2)$ is a rational number, where $n_{1,2}$ are the refraction indexes.

The mathematical theory of Dirac points is complicated even in 1D [7], but their treatment in the analysis of 1D bulk-edge correspondence is straightforward. The two bands with an accidental crossing are simply merged into one, and the resultant topological parameter is equal to the sum of the two respective parameters of the merged bands [7, Theorem 5.6]. Apart from that amendment, the rest of the theory – in particular, (32) – remain unchanged.

An alternative topological invariant. – Let N_m and P_m be the numbers of nulls and poles, respectively, below a gap m . Note that, at zero frequency, the mathematical impedance $\xi(0, 0)$ has a pole.¹ Further, one null and one pole of impedance occur necessarily at the two ends of any band gap. Thus, $P_m - N_m$ depends only on the type of the gap m . Namely, for an n -type gap $P_m - N_m = 0$, while for a p -type gap $P_m - N_m = 2$. Formally, $\gamma_m = (P_m - N_m)/2$. This quantity is a topological invariant, since any continuous change of physical and geometric parameters leading to a continuous change of the band structure (no gap closings) leaves γ_m unchanged. Its

¹At zero frequency, there is a distinction between the behavior of the mathematical and Bloch wave impedances. Since the electric field is constant, the mathematical impedance has a pole, while the electromagnetic impedance is finite and can be expressed via the effective constitutive parameters as $Z(0, 0) = \sqrt{\mu_{\text{eff}}/\epsilon_{\text{eff}}}$.

topological nature could be further illustrated by applying the argument principle and expressing γ_m via a contour integral of $\partial_\omega \log Z(0, \omega)$ in the complex plane.

Discussion and Conclusion. – We have considered the case of periodic lattices with mirror symmetry and no losses since, under these conditions, the topological principles come through most clearly. The impedance-centric view confers a transparent physical meaning on the bulk-boundary correspondence principle in 1D.

A systematic analysis of the behavior of Bloch wave impedance $Z(0, \omega)$ at the lattice cell boundary reveals several crucial features. First, within band gaps, the imaginary part $Z_I(0, \omega)$ decreases monotonically as a function of frequency either from a pole to a null or from a null to a pole. Further, when two materials with different behaviors abut and their band gaps overlap, the sum of their impedances must necessarily be zero at some frequency within that overlap range. An evanescent interface mode will then exist; otherwise it will not.

By a mnemonic analogy with the semiconductor terminology, we classify band gaps at the Γ and X points as type p or n depending on whether the Bloch wave impedance has a pole (p) or a null (n) at the bottom (lowest frequency) of that gap. An interface mode is shown to exist if and only if the two media have the opposite band gap types (one n and the other p). An alternative, and mathematically equivalent topological index is the difference between the numbers of poles and zeros of $Z(0, \omega)$ below a given band gap.

The above analysis applies to lossless media as a basic case. Our numerical experiments indicate that a small amount of loss does not change the conclusions qualitatively. For appreciable losses, all stages of the analysis need to be adjusted, as part of the rapidly expanding field of non-Hermitian topological photonics [6, 10, 11]. Abundant numerical evidence of bulk-boundary correspondence in 1D, including an example of lossy media, appears in [11]. We also expect the ideas of this letter to be extendable to more challenging problems in higher dimensions, with a variety of emerging applications [2, 6, 15–20].

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Appendix: Bloch impedance does not change sign within a band. – To prove this, suppose that the Bloch impedance $Z(0)$ has a zero at some frequency within a Bloch band. Then $(e_0 = 0, h_0 = 1)$ is an eigenvector of the monodromy matrix $T(a)$:

$$\begin{pmatrix} \alpha & i\beta \\ i\gamma & \alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence this matrix must have a Jordan form, $\beta = 0$. If so, $\alpha^2 = \det T(a) = 1$; $\Rightarrow \alpha = \pm 1$. Then $\lambda = \pm 1$ and $qa = 0, \pm\pi$. This shows that zero impedance can only occur at the band edges. Similarly, one can show that the Bloch impedance $Z(0)$ cannot have a pole within a gap. The constant sign of impedance follows by continuity.