Sub-Poissonian Statistics of Unidirectional Random Jumps on a Circumference

V. A. Markel

Optoelectronics, Instrumentation and Data Processing 1991, No.6, 109-112

Received April 29, 1991

Translated from:

Автометрия 1991, № 6, стр. 111-114

The sub-Poissonian statistics parameter ξ of unidirectional random jumps on a circumference is calculated. It is shown that ξ approaches -1 under certain conditions, which corresponds to very strong antibunching of events. The unidirectional jump model is considered as a direct analogy of step-wise multiphoton excitation in a multilevel system, and the results can be applied to the statistics of photons radiated from a particular level.

SUB-POISSONIAN STATISTICS OF UNIDIRECTIONAL RANDOM JUMPS ON A CIRCUMFERENCE

V.A.Markel

The sub-poissonian statistic parameter ξ of unidirectional random jumps on a circumference is calculated. ξ is shown to tend to -1 in certain conditions, this corresponds to a very strong antibunching of events. A unidirectional jump model is considered as applied to statistics of step-by-step excitation of multilevel systems, and to photon statistics, in particular.

The subject problem of random unidirectional jumps arises, in particular, in step-by-step excitation of multilevel systems with subsequent return to the ground state with energy release (photon emission, for example). Statistics of such events is known to be sub-poissonian, due to the dead-time effect [1–3]. The dead-time effect had been demonstrated experimentally to cause the sub-poissonian statistics [4] (antibunching of events, in particular).

Two directions of motion exist, as a rule, in real physical systems – toward energy increase (excitation) and energy decrease (decay). However, in order to prove the conceptual possibility of strong event antibunching, we limit our consideration to unidirectional jumps. Moreover, this problem is of importance from purely statistical grounds.

Let there be $L + 1$ points on a circumference, separated by equal arcs. Number them sequentially from zero to L. The particle undergoes instantaneous transitions from the point k to the point $k + 1$ ($k = 0, 1, ..., L - 1$), and from the point L to the point 0. The transition times are random, and the differential probability dp of the transition to occur in the time interval dt is given by $dp = \mu dt$ (μ is independent of time or the initial point of the transition. Thus the particle makes unit steps in the given direction along the circumference.

Let event A be recorded at the time when the particle moves from the point L to the point 0. Henceforth we will discuss the statistics of such elements. Let also certain initial conditions be set at $t = 0$, after which the system evolved for unlimited period of time. The process becomes steady-state at $t >> t_c = (L + 1)t_1$, where $t_1 = 1/\mu$ – the characteristic time of particle transition by a single step; and the probability to find a particle at any of the points becomes $p_\infty = 1/(L + 1)$.

Let us consider the time intervals $(t, t + T), t >> t_c$. Let $<N_T>$ be the average number of events A recorded with the period of time T, and $<\Delta N_T^2> = <N_T^2> - <N_T>^2$ be the variance of this number. The averaging may be done either over equal non-overlapping time intervals $(t, t + T)$, or over the ensemble of particles undergoing similar jumps. The sub-poissonian statistical parameter ξ_T is defined as follows [1, 2]:

$$<\Delta N_T^2> = <N_T> (1 + \xi_T). \tag{1}$$

The value of ξ_T characterizes the deviation of the statistics of events A from the Poissonian one. Depending on the sign of ξ_T, event bunching ($\xi_T > 0$), or antibunching ($\xi_T < 0$) takes place.

The ξ_T value against L was calculated in the present paper.

For $t >> t_c$, the following relations are valid:

$$<N_T> = JT, \quad J = \mu p_\infty;$$

$$<N_T^2> = \int_0^T \int [F(t_1 - t_2) + J\delta(t_1 - t_2)]dt_1 dt_2. \tag{2}$$

Here J is the steady-state probability of event A detection in a unit time, and the function $F(t_1 - t_2)$ is the joint probability of the event A occurring at times t_1 and t_2 (per $dt_1 dt_2$). To check the validity of (2) for $<N_T^2>$, let

us divide the time interval T into intervals Δt_i, they are so small that the probability to record the event A in any of them is much less than unity. N_T can be represented as a sum of contributions ΔN_i, and correspondingly N_T^2 as a double sum:

$$<N_T^2> = \sum <\Delta N_i \Delta N_j>.$$

At $i \neq j$, we get by definition of the function F $<\Delta N_i \Delta N_j> = F(t_i, t_j)\Delta t_i \Delta t_j$; and at $i = j$, $<\Delta N_i^2> = <\Delta N_i> = J\Delta t_i$. Passing from summation to integration, we obtain (2).

For the function F, obviously,

$$F(\tau) = F(-\tau) = J\mu p_L(\tau), \tag{3}$$

is valid, where $p_k(\tau)$ is the conditional probability to find the particle at the point k $(k = 0, 1, ..., L)$ at the instant of time $t = \tau$, if it was at the point 0 at $t = 0$. The function p_k satisfies the following conditions:

$$p_k(0) = \delta_{k0};\ \ p_k(\infty) = p_\infty;\ \ \sum_{k=0}^{L} p_k(\tau) = 1. \tag{4}$$

In view of (1)–(4), the equation for ξ_T takes the form:

$$\xi_T = \frac{2\mu}{T} \int_0^T dt \int_0^{T-t} d\tau [p_L(\tau) - p_\infty]. \tag{5}$$

The method of balance equations is the simplest approach to calculation of the functions p_k. It is based on the ensemble treatment. Let there be an ensemble of particles, for each of which the initial conditions $p_k(0) = \delta_{k0}$ were set. Then, treating $p_k(\tau)$ as a fraction of particles found at the point k at $t = \tau$, and proceeding from the balance of the processes, we obtain the following $L + 1$-dimensional linear set of differential equations:

$$\begin{cases} dp_0/dt = \mu(p_L - p_0), \\ dp_k/dt = \mu(p_{k-1} - p_k),\ \ k = 1, 2, ..., L. \end{cases} \tag{6}$$

The system (6) with the initial conditions (4) can be easily shown to have the solution

$$p_k(t) = p_\infty \sum_{j=0}^{L} z^{(L+1-k)j} \exp[(z^j - 1)\mu t],\ \ z = \exp\left[i\frac{2\pi}{L+1}\right]. \tag{7}$$

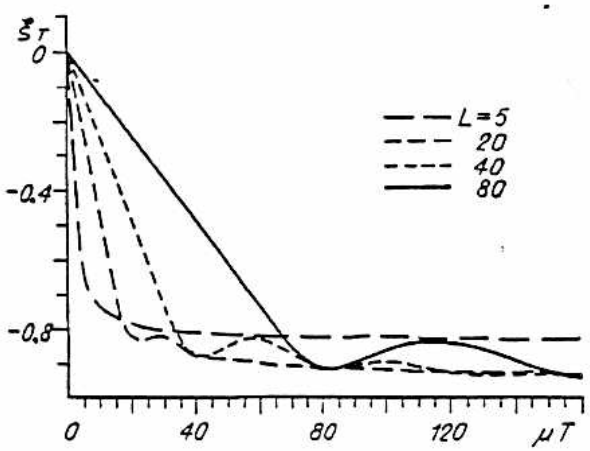
Obviously, all p_k are purely real values, the imaginary part of the sum (7) reduces to zero. The solution (7) may be conveniently rewritten in another form. Expanding $\exp[z^j]$ into a series and changing the order of summation, we obtain

$$p_k(t) = \sum_{m=0}^{\infty} \frac{\exp(-\mu t)(\mu t)^{k+(L+1)m}}{[k + (L+1)m]!}. \tag{8}$$

Thus the conditional probabilities p_k are represented by a sum of Poissonian distributions.

Expression (7) is more convenient for calculation of ξ_T. Note that for the most interesting case of $T >> t_c$, and, correspondingly, $<N_T> >> 1$, the integral (5) becomes substantially simpler, and we obtain

$$\xi_\infty = 2\mu \int_0^\infty [p_L(\tau) - p_\infty] d\tau = 2p_\infty \sum_{k=1}^{L} \frac{z^k}{1 - z^k} \simeq -\frac{L}{L+1}. \tag{9}$$



In this case $<\Delta N_T^2>/<N_T> = p_\infty$, and strong antibunching occurs at $L \to \infty$.

The result $\xi_\infty = -3/4$, corresponding to $L + 1 = 4$, was obtained in [2] for the fluorescence photons in a two-level atom excited by electromagnetic radiation under certain optimum conditions. Let us briefly discuss this result. Contrary to the particle of the present paper, which has a definite ("pure") state at any instant of time, an atom may have mixed states. As a result, the differential equation system for the density matrix of a two-level atom is in fact four-dimensional (two equations for the diagonal elements, and two for the real and imaginary parts of the off-diagonal element). The set of equations for the density matrix is similar to set (6) with $L = 3$, given the conditions imposed in [2] upon the internal times of the atom and its radiative excitation rate. These conditions yield the minimum possible value of $\xi_\infty = -3/4$. Generally speaking, atomic excitation and relaxations go through transient stages which correspond to the mixed states, these stages may be identified with the two additional points of the random jump model.

Similar to the above, we may predict for the step-wise excitation of a three-level atom or molecule a minimum value of $\xi_\infty = -7/8$. This value is achieved at no return from the first excited to the ground state.

In another limiting case of $T << t_c$ ($<N_T> << 1$), taking into account that $p_L(0) = 0$, we obtain

$$\xi_T = -JT = -<N_T>. \tag{10}$$

It is impossible to evaluate the sum resulting from integration of (7) according to (5) for arbitrary T. The figure presents numerical calculation results for various L. It is obvious that at large L the ξ_T plot against T has damped oscillations with a period t_c. ξ_T tends to ξ_∞ at $T >> t_c$.

REFERENCES

1. V.A.Markel and M.I.Stockman, *Optika i Spektroskopiya*, vol.65, iss. 6, 1988.
2. D.F.Smirnov and A.S.Troshin, *UFN*, vol. 153, iss. 2, 1987.
3. V.Vogel, *J. Phys. B.*, vol. 16, p. 4481, 1983.
4. E.Jakeman and J.H.Jefferson, *Opt. Acta*, vol: 33, p. 557, 1986.

29 April 1991

Novosibirsk

REFERENCES

1. V. A. Markel, M. I. Stockman, *Optika I Spektroskopiya* **165**, 1258 (1988).
2. D. F. Smirnov, A. S. Troshin, *UFN* **153**, 233 (1987).
3. V. Vogel, *J. Phys. B* **16**, 4481 (1983).
4. E. Jakeman, J. H. Jefferson, *Optica Acta* **33**, 557 (1986).