Balazs thought experiment and its implications for the electromagnetic force density in continuous media. Relativistic analysis

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ABSTRACT

We analyze the Balazs thought experiment under minimal assumptions and show that the two expressions for the force density that have been recently discussed in the literature (the conventional Lorentz force density and the generalized or Einstein-Laub force density) are fully consistent with conservation of energy, conservation of momentum and the center-of-energy theorem. This conclusion contradicts some of the previously published claims that the conventional Lorentz force density is inconsistent with the center-of-energy theorem. We identify the sources of errors in these claims. We conclude that the conventional Lorentz force density captures all experimentally-observable effects of interaction of electromagnetic fields with matter and no other law or postulate (such as a resolution of the so-called Abraham-Minkowski dilemma) is necessary.

1. Introduction

In the past ten years or so, there has been a spike of interest in the longstanding and somewhat obscure theoretical problem of determining the electromagnetic momentum density in continuous media [1–27]. The problem is usually referred to as the Abraham-Minkowski dilemma or controversy since these two authors have put forth, more than a hundred years ago, two competing expressions for the momentum density of light in material media. A historical perspective and review have been given by Milonni and Boyd [28] where many more relevant references published prior to 2010 can be found.

However, as was pointed out by Ginzburg and Ugarov in 1976 [29], the energy-momentum tensor is an auxiliary quantity, which can not be measured directly in an experiment. What is actually measured are force, pressure, velocity or displacement. To predict any of these quantities theoretically, it suffices to know the total force or the density of force induced by the electromagnetic fields. Correspondingly, we will be concerned in this paper with the density of electromagnetic force in continuous media.

The same approach was also adopted in a series of papers by Mansuripur (in some cases, with Zakharian) [30–39]. Stated concisely, Mansuripur argues that the density of force acting on a continuous medium can not be deduced from first principles but must be postulated. Once this is done, the different postulates must be put to test by investigating whether they are in agreement with the known conservation laws. Then the set of postulates that is found to be consistent with these laws will be the winner.

In this spirit, two competing expressions for the force density have been examined by Mansuripur. The first expression is based on the assumption that magnetization of macroscopic continuous media is caused by an electric current $c\nabla\times\mathbf{M}$, while the second expression is based on the assumption that magnetization is caused by displacement of magnetic monopoles and the associated magnetic current $\partial\mathbf{M}/\partial t$. In the first case, electromagnetic fields exert the conventional Lorentz force density on continuous media. In the second case, a different expression is obtained; Mansuripur refers to it as to the generalized Lorentz or Einstein-Laub force [30–34, 37]. We note however that the two competing expressions are special cases of the same generalized Lorentz force formula that includes both electric and magnetic charges and currents [Eq. (4) below].

It should be clarified that Mansuripur did not invoke magnetic monopoles directly. In fact, he has investigated the possibility of deriving the Einstein-Laub force by considering Amperian current loops [30], where, at one point, a

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creative argument was used to replace the field **B** by the field **H** in a certain mathematical expression ("we exclude this part of **B** from exerting a force on its own progenitor"). The result was still not quite the same as the Einstein-Laub force advocated by Mansuripur and, in subsequent publications, this line of reasoning was dropped. Instead, a point of view was adopted according to which various expressions for the force density are fundamental postulates of the macroscopic classical electrodynamics, which can be verified only experimentally.

Unfortunately, performing an experimental test of this kind is a difficult task. Apart from the fact that optical forces are very weak [1], both expressions for the force density investigated by Mansuripur yield the same predictions for the physical quantities measured in a typical experiment. However, this observation concerns either the time-averaged momentum transfer rate for a steady-state illumination or the total transferred momentum for a transient pulse. These are integral characteristics that gloss over the details of the force density on small time scales. An interesting fact is that these details are, in principle, experimentally observable. In 1953, Balazs has proposed an elegant thought experiment involving transient pulses [40], which is sensitive to some details of the force density other than just the time integral. The experiment is of the thought variety primarily because the effect it relies upon is extremely small. However, at least in principle, the effect is measurable and, therefore, we can consider it theoretically.

The Balazs thought experiment involves a transient pulse of light propagating through a slab of material (the "Balazs block") and the motion of all parts of the system is analyzed using some expression for the force density. One then checks whether the expression used is consistent with conservation of momentum and with linear uniform motion of the center of energy of the system "pulse+block". The thought experiment was originally formulated for a completely transparent and non-reflecting slab whose refractive index is different from unity. As these requirements are difficult to reconcile, Balazs has considered three special cases in which reflection is negligible. In the first case, an anti-reflection coating is used. This is not very satisfying because additional, unaccounted for forces can be generated inside the coating. The second approach is to send a p-polarized beam at the Brewster angle. This set-up involves no uncontrolled approximations but does not allow one to introduce nontrivial magnetic properties of the block. The third approach is to use a normally-incident pulse on a non-dispersive slab, which is impedance-matched to free space, that is, whose permittivity ϵ and permeability μ are equal and positive, $\epsilon = \mu > 0$. This is, arguably, the most interesting case.

In Ref. [37], Mansuripur has used the Balazs set-up with $\epsilon = \mu > 0$ to test the two above-mentioned expressions for the force density. A rectangular quasi-monochromatic incident pulse was considered. The quasi-monochromaticity allowed Mansuripur to assume that ϵ and μ are not affected by frequency dispersion. The conclusion reached was that the Einstein-Laub force is consistent with the law of uniform motion of the center of energy while the conventional Lorentz force is not. In the most recent publication in the series [39], Mansuripur suggested that, after all, the two force densities produce the same total force and torque on an isolated body but the conventional Lorentz force requires introduction of the hidden momentum, a theoretical concept to which Mansuripur objects. Since hidden momentum is not defined in [39], it is hard to evaluate the specifics of these claims. However, we will show that the two expressions for the force density do result in different overall forces, not just in different internal distributions of force. Also, the hidden momentum is physically related to the phenomenological forces of constraint [41] (which, essentially, hold a rigid body together) and not to the form of the electromagnetic force density. We will further show that, in spite of the two electromagnetic force densities being different, they are both consistent with all conservation laws including the center-of-energy theorem without invocation of the hidden momentum.

The Balazs thought experiment was further investigated by Chau and Lezec [42, 43]. In these works, the rather unrealistic assumption of zero absorption and reflection was removed. Due to the increased complexity, the problem was solved numerically for several incident quasi-monochromatic pulses using the FDTD method and some simple single-resonance formulas to describe the frequency dispersion of the constitutive parameters. This approach allowed Chau and Lezec to account for absorption, and an important message of Refs. [42, 43] is that, in order to demonstrate the center-of-energy theorem, it is important to account for energy transfer from the electromagnetic field to the slab. Similarly to Mansuripur, Chau and Lezec concluded that the Einstein-Laub force is consistent with all the conservation laws while the conventional Lorentz force is not.

The goal of this paper is three-fold. First, we will show that the Balazs thought experiment can be analyzed theoretically without making any of the approximations mentioned above. We will make only minimal assumptions about the slab, namely, that it is homogeneous in the lateral direction, non-chiral and therefore does not rotate the

[1] This paper, as well as many papers cited above, are primarily concerned with high-frequency electromagnetic phenomena. Forces in low-frequency electromagnetic machines such as electric motors can, of course, be large. However, it is hard to imagine how the Balazs thought experiment can be set up for, say, 60Hz electromagnetic waves.

plane of polarization [2], and not amplifying. The incident pulse can be quite arbitrary and not necessarily (quasi)-monochromatic. We will only require that the pulse be transient. A more precise mathematical condition of transiency will be formulated below. No assumptions will be made about the constitutive relations in the slab. In particular, these relations can be spatially- and temporally-dispersive or even nonlinear. The slab is not required to be homogeneous in the depth direction and can be structured or otherwise change its properties. In spite of this very general formulation, we will prove all conservation laws analytically without resorting to numerical simulations.

The second goal is to show that one of the conclusions reached by Mansuripur in [37] and by Chau and Lezec in [43] is incorrect. Namely, the conventional Lorentz force density is fully consistent with all conservation laws including the center-of-energy theorem. The reasons why the incorrect conclusions were reached previously are as follows. Mansuripur failed to realize that, even in the case of non-absorbing slabs, there can be non-zero energy transfer from the electromagnetic field to the slab and this energy transfer affects the center of energy position after all the fields die out. In fact, Mansuripur would have obtained a violation of the center-of-energy theorem even for the Einstein-Laub force had he considered an absorbing slab. In the work by Chau and Lezec, energy transfer was taken into account albeit numerically and therefore imprecisely. However, these two authors have used an incorrect expression for the conventional Lorentz force density (i.e., see Appendix A in [43]). Apparently, Chau and Lezec made the same mistake as was made more recently by Torchigin [44], who took the term "displacement current" too literally and posited that the induced electric current in the conventional Lorentz force formula is $\partial \mathbf{D}/\partial t$ rather than $\partial \mathbf{P}/\partial t$. This approach predicts, in particular, a non-zero force acting on empty space.

The third goal of this paper is to finally put the Abraham-Minkowski controversy to rest. Indeed, we will show that, as soon as one decides on the model for magnetization of continuous media (Amperian current loops or displaceable magnetic monopoles), the ensuing expression for the electromagnetic force density is sufficient to predict any experimentally-measurable effect of interaction of electromagnetic fields with matter. No other law or postulate is needed to this end and, in particular, a resolution to the Abraham-Minkowski dilemma is also not needed. Moreover, both expressions for the electromagnetic force density are perfectly consistent with all known conservation laws and therefore it is not possible to determine which of the two is correct *theoretically*.

Many papers on the Abraham-Minkowski dilemma revolve around the question whether the so-called Abraham's force [Eq. (31) below] is "real" or measurable. In this respect, we can note the following. It will be shown below that the terms involving a time derivative are present in all expressions that follow from the generalized force law (4). These terms are reducible to the conventional Abraham's force (31) only upon making some assumptions about the medium and choosing a particular model for magnetization. Accounting for the Abraham-like terms results in zero total momentum transfer from field to matter in any transient or stationary process. However, these terms do result in a shift of the center of energy of the system, and therefore they are "real" and experimentally-measurable, at least in principle. But arriving at this conclusion does not require any knowledge about the electromagnetic momentum density in continuous media, if this quantity can even be defined with sufficient generality.

The remainder of this paper is organized as follows. In Section 2, we discuss the theoretical underpinnings of the two competing expressions for the force density that were investigated by Mansuripur. We show that these expressions are direct consequences of the underlying model for magnetization of matter. In the First (Amperian) Model, magnetization is caused by electric currents. In the Second Model, magnetization is caused by displacement of magnetic monopoles. In this Section, we also discuss how the "conventional" force density that appears in many textbooks or research papers (including the conventional Abraham's force term) can be derived from a more general expression under the assumption of local, linear and non-dispersive medium. In Section 3, we describe the Balazs thought experiment and the various approximations involved. In Section 4, we relate mathematically the displacement of the center-of-energy of the slab to the first time-moment of the force. In Section 5, we adduce the relevant mathematical properties of the incident, reflected and transmitted pulses. Section 6 is central to this paper. Here we prove that the three laws (conservation of energy, conservation of momentum and center-of-energy theorem) hold for both expressions for the force density. However, to achieve this result, we must properly account for energy transfer from the electromagnetic field to the slab. Details of this transfer are different in the two models of magnetization but the end result is the same: all conservation laws are satisfied regardless of which magnetization model and, correspondingly, which expression for the force density is used. Finally, Section 7 contains a summary and a discussion of the obtained results.

Gaussian system of units is used throughout this paper.

[2]This assumption is needed to avoid the consideration of rotational motion.

2. Two expressions for the electromagnetic force density in continuous media

It is now abundantly clear that in non-magnetic media all forces are unambiguously derivable from the Lorentz force density acting on the electric current $\partial \mathbf{P}/\partial t$ and the electric charge $-\nabla \cdot \mathbf{P}$, where $\mathbf{P}$ is the vector of electric polarization. Here the term $\partial \mathbf{P}/\partial t$ contains *all* induced electric currents induced, including the conductivity current if the medium is conducting. There is no need to write the term $\sigma \mathbf{E}$ separately although doing so is not a mistake for linear and local conductors [45]. The expression $\partial \mathbf{P}/\partial t$ is however more general and encompasses the cases of nonlinear and nonlocal media. Formulas that do not follow from the Lorentz force law still appear in the literature, even recently, i.e., by Torchigin [44]. However, these formulas rely on incorrectly used approximations [3] or involve *ad hoc* arguments [4].

The confusion around applicability of the Lorentz force density to continuous media grows, at least in part, from the misguided tradition of distinguishing between the so-called bound and free charges and currents and the attempts to apply different laws of motion to these components of the total current and charge. In fact, the early discussion has been largely focused on the question whether the term $\partial \mathbf{P}/\partial t$ represents a "real" current and whether the usual Lorentz force can be applied to it. Ginzburg and Ugarov argued that, from the theoretical point of view, there is no basis to treat this term differently from the conductivity current [29]. This conclusion was confirmed experimentally by Walker *et al.* [46–48].

However, an uncertainty in the definition of the force exists if the medium has nontrivial magnetic properties. The reason is that the classical expression for the Lorentz force tells us nothing about particles with intrinsic magnetic moments. Indeed, the Lorentz force formula contains the electric charge and current and, in the generalized form (4) (see below), also the magnetic charge and current, but no magnetic dipoles or any other multipole moments. If we want to compute the force acting on a particle with, say, an electric dipole moment, we must devise a model representing the dipole as a distribution of electric charge [5]. The same problem arises if a particle has an intrinsic magnetic moment, and in this case it is less obvious what the "microscopic" model for the magnetic dipole should be. The papers by Mansuripur [30–39] and Chau and Lezec [42, 43] were largely aimed at resolving the above ambiguity.

In the case of an electric dipole, the situation is relatively simple. There exist detailed and experimentally verified models (both classical and quantum) explaining the electric polarization of small macroscopic particles, molecules or atoms in terms of electric charge distribution. In particular, it follows from all these models that the force acting on a small particle with the total electric dipole moment $\mathbf{d}$ in an external electric field $\mathbf{E}$ is $(\mathbf{d} \cdot \nabla)\mathbf{E}$ [6]. What is important for us now is that application of the Lorentz force to particles with intrinsic electric dipole moments is straightforward and unambiguous since we know how these dipoles are made up microscopically. Moreover, even if the particle has an intrinsic magnetic moment due to the classical current (or the orbital current in the quantum case), one can still apply the classical Lorentz force directly: we simply substitute the current and charge densities into the formula and integrate over the volume. There are no ambiguities present in this calculation.

The uncertainty arises when we include into consideration the intrinsic magnetic moments of elementary particles [7]. Indeed, we do not have a good microscopic model for how these moments are made up. One can imagine that they are created by some persistent electric current (so-called Amperian current loops). One can as well assume that the intrinsic magnetic moments are made up as bound states of two opposite magnetic charges (monopoles), even though the latter

[3]For example, in the beginning of Section 4 of Ref. [44], Torchigin writes "...the pressure produced by a light wave during normal incidence on the boundary between free space and the optical medium according to any of the equations (3),(4) and (5) [of [44] - V.M.] is equal to zero ...because the phase shift between dP/dt and H is equal to $\pi/2$." But this phase shift is $\pi/2$ only if the refractive index of the medium is real. In this case, one must *either* consider the far edge of the slab and the wave reflected from it, which will result in the phase shift being $\neq \pi/2$ *or otherwise* consider a transient pulse entering a semi-infinite medium. At the front edge of this pulse, time-averaging does not yield the force density correctly. In either scenario, the total force on the slab is nonzero. In a weakly absorbing half-space, the Lorentz force density is small but, when integrated over the (slowly-decaying) transmitted wave, it yields the total pressure that is equal to the sum of the incident and the reflected waves momenta per unit time per unit surface, just as is expected from momentum conservation.

[4]To fix the perceived problem with the Lorentz force density (which in reality does not exist), Torchigin writes "The expression for such interaction should be corrected as follows. First, the current is equal to $d\mathbf{D}/dt$ rather than $d\mathbf{P}/dt$." No justification is given by Torchigin for why the current should be $\partial \mathbf{D}/\partial t$ except for an *ad hoc* argument involving a capacitor. Namely, Torchigin suggests incorrectly that an electric current can flow through the capacitor even if it is filled with vacuum. In reality, the electric current is given by the first equation in (2b) below and is identically zero in regions of space where there are no electric charges. The force density proposed by Torchigin can be nonzero in vacuum and Torchigin does not explain to *what* the proposed force is applied in this case.

[5]We can as well represent an electric dipole as a magnetic current loop, but this is usually not done.

[6]For this reason, it is frequently assumed that the electric part of the force density in polarized dielectrics "consisting of dipoles" is $(\mathbf{P} \cdot \nabla)\mathbf{E}$. If we restrict consideration to rigid bodies, the difference between this expressions and $-(\nabla \cdot \mathbf{P})\mathbf{E}$ is insignificant. A more detailed discussion of the two expressions and of various other forms appearing in the literature is given in Section 2.4 below.

[7]The intrinsic electric dipole and electric quadrupole moments of the stable elementary particles are zero, at least, in the Standard Model. Therefore we do not discuss these multipole moments here.

have not been observed experimentally in a free state. It is not clear why the elementary particles made of opposing magnetic charges would not also have an electric dipole moment, since rotating magnetic monopoles produce just that, and electrons and protons are known to have nonzero mechanical angular momenta. However, one is free to dismiss this inconsistency as something yet unexplored. In Refs. [30–39], Mansuripur makes one step further and postulates that the *macroscopic* magnetization, i.e., of a piece of iron, can also be caused by displacement of magnetic monopoles (to be more precise, this is one of two competing postulates invoked by Mansuripur).

Therefore, at the heart of the discussion in Refs. [30–39] lie two different and competing physical models for magnetization of matter. The First Model is based on the premise that magnetic fields can be caused only by currents of electric charge. According to this model, a statically magnetized object supports a steady, non-dissipating current of electric charge $c\nabla\times\mathbf{M}$, which is the source of the magnetic field produced by the magnet. An external magnetic field exerts the usual Lorentz force density on this current. This model is known as Amperian.

The Second Model assumes that magnetization is caused by displacement of magnetic monopoles, similarly to the electric polarization being caused by displacement of electric charges. This point of view is rather old and it possesses a strong intuitive appeal for a variety of reasons. Chief among these is simplicity, symmetry and transparency of various mathematical expressions obtained in this model. Another apparent reason is that this model seems to allow more freedom in deciding which types of media can exist in nature, while the Amperian model is more restrictive [49]. Of course, we should keep in mind that the Second Model applies only to magnetization due to alignment of the intrinsic magnetic moments of elementary particles; it does not apply to magnetic effects in composites and metamaterials.

We will show momentarily how the two expressions for the force density can be derived within the two different models for magnetization.

2.1. Maxwell's equations in the presence of electric and magnetic charges

It is instructive to start by writing out the microscopic Maxwell's equations for the case when both electric and magnetic charges can be present. Consider point particles of rest masses m_i, electric charges q_i and magnetic charges g_i moving in electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ along the classical trajectories $\mathbf{r}_i(t)$ (by "classical" we mean relativistic but not quantum). The microscopic Maxwell's equations read in this case

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\mathrm{e}}\ , \ \ \nabla\cdot\mathbf{B} = 4\pi\rho_{\mathtt{m}}\ , \ \ -\nabla\times\mathbf{E} = \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\mathtt{m}}\ , \ \ \nabla\cdot\mathbf{E} = 4\pi\rho_{\mathrm{e}}\ , \tag{1}$$

where

$$\rho_{\mathrm{e}}(\mathbf{r},t) = \sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i(t))\ , \qquad \rho_{\mathtt{m}}(\mathbf{r},t) = \sum_i g_i\delta(\mathbf{r}-\mathbf{r}_i(t))\ , \tag{2a}$$

$$\mathbf{J}_{\mathrm{e}}(\mathbf{r},t) = \sum_i q_i\dot{\mathbf{r}}_i(t)\delta(\mathbf{r}-\mathbf{r}_i(t))\ , \qquad \mathbf{J}_{\mathtt{m}}(\mathbf{r},t) = \sum_i g_i\dot{\mathbf{r}}_i(t)\delta(\mathbf{r}-\mathbf{r}_i(t))\ . \tag{2b}$$

are the densities and currents of the electric (indexed by the letter e) and magnetic (indexed by m) charge. Overhead dot denotes time differentiation. It is easy to see that the charges and currents defined in (2) satisfy the continuity equations

$$\frac{\partial\rho_{\mathrm{e}}}{\partial t} + \nabla\cdot\mathbf{J}_{\mathrm{e}} = 0\ , \ \ \frac{\partial\rho_{\mathtt{m}}}{\partial t} + \nabla\cdot\mathbf{J}_{\mathtt{m}} = 0\ . \tag{3}$$

Note that the divergence equations in (2.1) are not independent but follow from the two curl equations and the continuity equations (3).

The density of the Lorentz force $\mathbf{f}$ is given in this general setting by

$$\mathbf{f} = \rho_{\mathrm{e}}\mathbf{E} + \rho_{\mathtt{m}}\mathbf{B} + \frac{1}{c}\left(\mathbf{J}_{\mathrm{e}}\times\mathbf{B} - \mathbf{J}_{\mathtt{m}}\times\mathbf{E}\right)\ , \tag{4}$$

where dependence of all quantities on $\mathbf{r}$ and t is implied. We have chosen the signs in the above expression for complete symmetry between the electric and the magnetic phenomena. Thus, two magnetic charges of the same sign experience a repulsive force and two parallel wires carrying magnetic current in the same direction experience an attractive force.

To introduce a continuous medium, we make some assumption about the currents. Namely we assume that all currents inside the medium are induced by the fields and satisfy all relevant symmetry properties. We can write quite generally

$$\mathbf{J}_{\mathrm{e}} = \frac{\partial\mathbf{X}_{\mathrm{e}}}{\partial t} + c\nabla\times\mathbf{Y}_{\mathrm{e}}\ , \ \ \mathbf{J}_{\mathtt{m}} = \frac{\partial\mathbf{X}_{\mathtt{m}}}{\partial t} + c\nabla\times\mathbf{Y}_{\mathtt{m}}\ . \tag{5}$$

where $\mathbf{X}_e(\mathbf{r},t)$, $\mathbf{Y}_e(\mathbf{r},t)$, $\mathbf{X}_m(\mathbf{r},t)$, $\mathbf{Y}_m(\mathbf{r},t)$ are some functionals of the fields and are supported inside the medium (and zero in vacuum).

2.2. The First Model

In the First Model, we assume that there are no magnetic charges and currents and, specifically, make the following choice for the quantities appearing in the right-hand sides of (5):

$$\mathbf{X}_e = \mathbf{P}\,, \quad \mathbf{Y}_e = \mathbf{M}\,, \quad \mathbf{X}_m = 0\,, \quad \mathbf{Y}_m = 0\,, \tag{6}$$

Here $\mathbf{P}$ and $\mathbf{M}$ are usually referred to as the vectors of polarization and magnetization. We denote the fields that one computes in the First Model by $\mathbf{E}_1$ and $\mathbf{B}_1$. Then Maxwell's equations acquire the conventional form

$$\nabla\times\mathbf{B}_1 = \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi}{c}\mathbf{J}_e\,, \quad \nabla\cdot\mathbf{B}_1 = 0\,, \quad -\nabla\times\mathbf{E}_1 = \frac{1}{c}\frac{\partial \mathbf{B}_1}{\partial t}, \quad \nabla\cdot\mathbf{E}_1 = 4\pi\rho_e\,, \tag{7}$$

where

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} + c\nabla\times\mathbf{M}\,, \quad \rho_e = -\nabla\cdot\mathbf{P}\,. \tag{8}$$

Introducing the auxiliary fields

$$\mathbf{D}_1 = \mathbf{E}_1 + 4\pi\mathbf{P}\,, \quad \mathbf{H}_1 = \mathbf{B}_1 - 4\pi\mathbf{M}\,, \tag{9}$$

we can rewrite (7) identically as

$$\nabla\times\mathbf{H}_1 = \frac{1}{c}\frac{\partial \mathbf{D}_1}{\partial t}\,, \quad \nabla\cdot\mathbf{B}_1 = 0\,, \quad -\nabla\times\mathbf{E}_1 = \frac{1}{c}\frac{\partial \mathbf{B}_1}{\partial t}\,, \quad \nabla\cdot\mathbf{D}_1 = 0\,. \tag{10}$$

As follows from (4), the force density in the First Model is given by

$$\mathbf{f}_1 = \rho_e\mathbf{E}_1 + \frac{1}{c}\mathbf{J}_e\times\mathbf{B}_1\,. \tag{11}$$

We now substitute $\mathbf{J}_e$ from the first equation in (7) into (11). This yields

$$\mathbf{f}_1 = \rho_e\mathbf{E}_1 + \frac{1}{4\pi}\left[\left(\nabla\times\mathbf{B}_1\right)\times\mathbf{B}_1 - \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t}\times\mathbf{B}_1\right]\,. \tag{12}$$

By using the identity

$$\frac{\partial \mathbf{E}}{\partial t}\times\mathbf{B} = \frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t} - \mathbf{E}\times\frac{\partial \mathbf{B}}{\partial t}$$

and the third equation in (7), we can write $\mathbf{f}_1$ in a more symmetric form:

$$\mathbf{f}_1 = \rho_e\mathbf{E}_1 + \frac{1}{4\pi}\left[(\nabla\times\mathbf{B}_1)\times\mathbf{B}_1 + (\nabla\times\mathbf{E}_1)\times\mathbf{E}_1 - \frac{1}{c}\frac{\partial(\mathbf{E}_1\times\mathbf{B}_1)}{\partial t}\right]\,. \tag{13}$$

2.3. The Second Model

In the Second Model, the medium can contain both electric and magnetic charges and currents. More specifically, the following choices are made for the quantities appearing in the right-hand sides of (5):

$$\mathbf{X}_e = \mathbf{P}\,, \quad \mathbf{Y}_e = 0\,, \quad \mathbf{X}_m = \mathbf{M}\,, \quad \mathbf{Y}_m = 0\,. \tag{14}$$

Compare this to (6). Now, instead of (7), we have

$$\nabla\times\mathbf{B}_2 = \frac{1}{c}\frac{\partial \mathbf{E}_2}{\partial t} + \frac{4\pi}{c}\mathbf{J}_e\,, \quad \nabla\cdot\mathbf{B}_2 = 4\pi\rho_m\,, \quad -\nabla\times\mathbf{E}_2 = \frac{1}{c}\frac{\partial \mathbf{B}_2}{\partial t} + \frac{4\pi}{c}\mathbf{J}_m\,, \quad \nabla\cdot\mathbf{E}_2 = 4\pi\rho_e\,, \tag{15}$$

where

$$\mathbf{J}_{\rm e} = \frac{\partial \mathbf{P}}{\partial t} \ , \quad \rho_{\rm e} = -\nabla \cdot \mathbf{P} \ , \quad \mathbf{J}_{\rm m} = \frac{\partial \mathbf{M}}{\partial t} \ , \quad \rho_{\rm m} = -\nabla \cdot \mathbf{M} \ , \tag{16}$$

and $\mathbf{E}_2, \mathbf{B}_2$ are the fields that are obtained by solving Maxwell's equations in the Second Model. As above, we introduce the auxiliary quantities

$$\mathbf{D}_2 = \mathbf{E}_2 + 4\pi \mathbf{P} \ , \ \ \mathbf{H}_2 = \mathbf{B}_2 + 4\pi \mathbf{M} \ . \tag{17}$$

Note the different sign in the second equation as compared to (9). Then the macroscopic Maxwell's equations of the Second Model acquire the form

$$\nabla\times\mathbf{B}_2 = \frac{1}{c}\frac{\partial \mathbf{D}_2}{\partial t} \ , \ \ \nabla\cdot\mathbf{H}_2 = 0 \ , \ \ -\nabla\times\mathbf{E}_2 = \frac{1}{c}\frac{\partial \mathbf{H}_2}{\partial t} \ , \ \ \nabla\cdot\mathbf{D}_2 = 0 \ . \tag{18}$$

The force density in the Second Model is

$$\mathbf{f}_2 = \rho_{\rm e}\mathbf{E}_2 + \rho_{\rm m}\mathbf{B}_2 + \frac{1}{c}(\mathbf{J}_{\rm e}\times\mathbf{B}_2 - \mathbf{J}_{\rm m}\times\mathbf{E}_2) \ . \tag{19}$$

Using the two curl equations in (15) to express $\mathbf{J}_{\rm e}$ and $\mathbf{J}_{\rm m}$ and applying the same transformations as above, we obtain

$$\mathbf{f}_2 = \rho_{\rm e}\mathbf{E}_2 + \rho_{\rm m}\mathbf{B}_2 + \frac{1}{4\pi}\left[(\nabla\times\mathbf{B}_2)\times\mathbf{B}_2 + (\nabla\times\mathbf{E}_2)\times\mathbf{E}_2 - \frac{1}{c}\frac{\partial(\mathbf{E}_2\times\mathbf{B}_2)}{\partial t}\right] \ . \tag{20}$$

2.4. Summary of forces

The two forces (13) and (20) look very similar except for the term $\rho_{\rm m}\mathbf{B}_2$, but are not the same because the fields in the two models are not the same. However, by comparing (10) to (18) and requiring that $\mathbf{M}$ and $\mathbf{P}$ be the same in both cases, we can establish the following correspondence

$$\mathbf{E}_2 = \mathbf{E}_1 \ , \ \ \mathbf{D}_2 = \mathbf{D}_1 \ , \ \ \mathbf{B}_2 = \mathbf{H}_1 \ , \ \ \mathbf{H}_2 = \mathbf{B}_1 \ . \tag{21}$$

Thus, both models predict the same electric field and electric induction. However, while the First Model predicts the magnetic field $\mathbf{B}_1$ and the auxiliary field $\mathbf{H}_1$, the Second Model predicts the magnetic field $\mathbf{B}_2 = \mathbf{H}_1$ and the auxiliary field $\mathbf{H}_2 = \mathbf{B}_1$. So the fields $\mathbf{B}$ and $\mathbf{H}$ of the two models are cross-related.

As a practical matter, the equations (18) of the Second Model are almost never solved directly. In fact, this is not necessary; if one wishes to use the Second Model, it suffices to solve the conventional equations (10) of the First Model and then use the correspondence rules (21). For this reason, many textbooks refer to **B** as to the magnetic induction and to **H** as to the magnetic field. This convention is a legacy of 19-th century physics; at that time, the Second Model of magnetization seemed as physically plausible as anything else. For the same reason, many physicists tend to think of **H** as of the "real" magnetic field, which should appear in the force- and energy-related expressions such as the Lorentz force, the Poynting vector, etc. However, if we accept the First Model as the underlying physical principle, then the true magnetic field is **B**, at least in the sense that it should appear in the definitions of the above physical quantities.

We thus conclude that the following approach is mathematically equivalent to considering the two models explicitly and solving the corresponding sets of equations (10) and (18) separately. We will *always* use the conventional (First Model) form of Maxwell's equations (10) to compute the fields, and we will write simply $\mathbf{E} = \mathbf{E}_1$ and $\mathbf{B} = \mathbf{B}_1$. We will refer to these quantities as to the electric and magnetic fields. If we wish to use the Second Model, we will rely on the correspondence rules given in (21). Then the force densities in the two models considered above are given by the following expressions:

$$\mathbf{f}_1 = \rho_{\rm e}\mathbf{E} + \frac{1}{4\pi}\left[(\nabla\times\mathbf{B})\times\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{E} - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}\right] \ , \tag{22a}$$

$$\mathbf{f}_2 = \rho_{\rm e}\mathbf{E} + \rho_{\rm m}\mathbf{H} + \frac{1}{4\pi}\left[(\nabla\times\mathbf{H})\times\mathbf{H} + (\nabla\times\mathbf{E})\times\mathbf{E} - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t}\right] \ . \tag{22b}$$

Since the Poynting vector is given in the two models by the expressions

$$\mathbf{S}_1 = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \ , \ \ \mathbf{S}_2 = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} \ , \tag{23}$$

we can also write

$$\mathbf{f}_1 = \rho_{\mathrm{e}}\mathbf{E} + \frac{1}{4\pi}\left[(\nabla\times\mathbf{B})\times\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{E}\right] - \frac{1}{c^2}\frac{\partial\mathbf{S}_1}{\partial t} \ , \tag{24a}$$

$$\mathbf{f}_2 = \rho_{\mathrm{e}}\mathbf{E} + \rho_{\mathrm{m}}\mathbf{H} + \frac{1}{4\pi}\left[(\nabla\times\mathbf{H})\times\mathbf{H} + (\nabla\times\mathbf{E})\times\mathbf{E}\right] - \frac{1}{c^2}\frac{\partial\mathbf{S}_2}{\partial t} \ . \tag{24b}$$

2.5. Discussion of forces

Expressions (24) are mathematically equivalent to those used by Mansuripur, for example, in [39] (compare to equations 12 and 15 of that reference), and by many other authors, modulo the expression $\rho_{\mathrm{e}}\mathbf{E}$ and $\rho_{\mathrm{m}}\mathbf{H}$ instead of $(\mathbf{P}\cdot\nabla)\mathbf{E}$ and $(\mathbf{M}\cdot\nabla)\mathbf{H}$. This difference is insignificant if one wishes to describe the motion of a rigid body and, moreover, in the set-up of the Balazs thought experiment, these terms are identically zero. However, in order to establish correspondence between (24) and various expressions appearing in the literature, we provide below a more detailed discussion.

2.5.1. Electric part of the Lorentz force

The term $\rho_{\mathrm{e}}\mathbf{E}$ is the same in both models of magnetization and it appears in both equations in (24). With the use of (8) or (16), we can write

$$\rho_{\mathrm{e}}\mathbf{E} = -(\nabla\cdot\mathbf{P})\mathbf{E} \equiv \mathbf{f}_{\mathrm{a}} \ . \tag{25a}$$

While the above expression does not involve any *ad hoc* assumptions, an alternative expression

$$(\mathbf{P}\cdot\nabla)\mathbf{E} \equiv \mathbf{f}_{\mathrm{b}} \tag{25b}$$

is frequently used in place and instead of $\mathbf{f}_{\mathrm{a}}$. We note right away that

$$\int \mathbf{f}_{\mathrm{a}} d^3r = \int \mathbf{f}_{\mathrm{b}} d^3r \ , \ \ \int \mathbf{r}\times\mathbf{f}_{\mathrm{a}} d^3r = \int \mathbf{r}\times\mathbf{f}_{\mathrm{b}} d^3r \ , \tag{26}$$

where the integral is taken over any three-dimensional region containing the object. Therefore, the total force and torque are the same in the two cases and the forces $\mathbf{f}_{\mathrm{a}}$ and $\mathbf{f}_{\mathrm{b}}$ are physically indistinguishable for rigid bodies. In [50], (26) was demonstrated for a particular electrostatic problem. However, the equalities hold generally as long as $\mathbf{P}(\mathbf{r})$ vanishes outside of the object. Note that, for (26) to be true, $\mathbf{E}$ is not required to be irrotational and the relation between $\mathbf{P}$ and $\mathbf{E}$ is not required to be linear. The force $\mathbf{f}_{\mathrm{b}}$ is often preferred because it relates better to the mental picture of a medium as a collection of dipoles. However, a continuous medium is not equivalent to a collection of elementary dipoles and $\mathbf{P}$ is not generally equal to the dipole moment per unit volume, except in dilute gases and some other special cases [51–56].

One potential source of ambiguity in the above expressions is that (25) can contain products of delta-functions and discontinuous function. For example, consider a boundary $z = 0$ between a vacuum and a dielectric medium at which the normal component of the electric field is discontinuous. On the other hand, the interface supports an electric charge density, which is described mathematically by the delta-function $\delta(z)$. One can ask, which electric field acts on this surface charge, the one just to the left or just to the right of the interface?

We can resolve the ambiguity by considering a finite transition layer of width Δ and then taking the limit $\Delta \to 0$. Let the permittivity satisfy the properties $\epsilon(z) = 1$ for $z < 0$ and $\epsilon(z) \to \epsilon_0$ when $z \to \infty$ and let the transition occur smoothly on the characteristic scale of Δ, which can be arbitrarily small. We further assume that the electric field is polarized along the Z-axis. The induced electric charge in this case is

$$\rho_{\mathrm{e}}(z) = -\frac{dP_z(z)}{dz} = -\frac{d}{dz}\frac{\epsilon(z)-1}{4\pi}E_z(z) \ , \tag{27}$$

We know from the condition $\nabla\cdot\mathbf{D}=0$ that $d[\epsilon(z)E_z(z)]/dz=0$. Therefore, we have for the z-component of the force density $\mathbf{f}_{\rm a}$:

$$f_{{\rm a},z}(z)=\frac{1}{4\pi}E_z(z)\frac{d}{dz}E_z(z)\ . \tag{28}$$

Integrating over z, we obtain for the pressure (force per unit surface):

$$\mathscr{F}_{{\rm a},z}\equiv\int f_{{\rm a},z}(z)dz=\frac{1}{8\pi}(E_\infty^2-E_0^2)\ , \tag{29}$$

where E_0 is the electric field in the region $z<0$ and $E_\infty=\lim_{z\to\infty}E_z(z)$. We know that $E_0=\epsilon_0E_\infty$ and also that $\sigma_{\rm e}\equiv\int\rho_{\rm e}(z)dz=(1/4\pi)(E_\infty-E_0)$. From this, we find

$$\mathscr{F}_{{\rm a},z}=\frac{\sigma_{\rm e}}{2}(E_\infty+E_0)=-\frac{E_0^2}{8\pi}\frac{\epsilon_0^2-1}{\epsilon_0^2}\ . \tag{30a}$$

Thus, the pressure is equal to the surface charge density $\sigma_{\rm e}$ times the average of the electric field on the two sides of the discontinuity.

If we use $\mathbf{f}_{\rm b}$ to compute the pressure, the result would be different. This is not surprising since we can only prove that the *total* force acting on a finite object is the same in the two cases, and here we did not consider the far face of the slab (if we did, the total force acting on the slab would be zero). Omitting straightforward calculations, the "b-type" pressure is

$$\mathscr{F}_{{\rm b},z}=-\frac{E_0^2}{8\pi}\frac{(\epsilon_0-1)^2}{\epsilon_0^2}\ . \tag{30b}$$

So, while the total force on any finite-width slab is always zero in the one-dimensional example considered here, the predicted induced *strain* depends on which force is used in the calculation, $\mathbf{f}_{\rm a}$ or $\mathbf{f}_{\rm b}$.

While the above examples are rather specific, they illustrate a general mathematical fact: in the formulas of the type (25), discontinuous functions can be safely integrated with delta-functions by defining their values as the arithmetic average of the left and right limits. The same approach was used in Ref. [50]. We note that, in [50], the total force exerted on a dielectric half-space was shown to be the same regardless of which force is used [8]. This is different from out finding as expressed in (30). The different conclusions can be easily reconciled by noting that the external field in Ref. [50] was produced by a point charge and the problem was not one-dimensional; in the set-up of [50], the force density at the far face of a sufficiently thick slab is insignificant and one can consider the half-space as a limiting case. In the one-dimensional geometry considered above, a half-space is never a good approximation since the force exerted on the far face of the slab is never negligible.

2.5.2. Nondispersive linear media and the Abraham force

Expressions in (24) do not contain what is known as the "Abraham's force". The latter is usually written in the form [29, 57]

$$\mathbf{f}_{\tt A}=\frac{\epsilon\mu-1}{4\pi c}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t}\ , \tag{31}$$

where ϵ and μ are the permittivity and permeability of the medium. Obviously, these quantities must be real-valued for (31) to make physical sense. The expressions (24) contain similar terms, $\partial(\mathbf{E}\times\mathbf{B})/\partial t$ and $\partial(\mathbf{E}\times\mathbf{H})/\partial t$, depending on the model for magnetization used. Just like the Abraham's force, these terms yield zero total momentum transfer from field to matter in any transient process and zero total time-averaged force for periodic or statistically-stationary fields. However, these terms produce a nonzero effect on the displacement of the center of energy, which is the nexus of the Balazs thought experiment.

[8] In the set-up of Ref. [50], the force density expression (25a) yields only a surface term and (25b) yields both surface and volume terms so that, strictly speaking, the two force densities are not the same. However, they produce the same total pull when integrated over a sufficiently deep sub-surface layer.

Although the terms $\partial(\mathbf{E}\times\mathbf{B})/\partial t$ and $\partial(\mathbf{E}\times\mathbf{H})/\partial t$ are obviously different from (31), we will show momentarily how the Abraham's force emerges from the expression (24b) under the assumption that the medium is linear, scalar, local and non-dispersive, or has the simple dispersion relation characteristic of conductors at low frequencies. Using these assumptions, we will obtain the same expression for the force density as appears, for example, in [57] and contains the conventional Abraham's force (31). We emphasize that the expression (24b) from which we will start is more general as it does not rely on any assumptions about the medium except that it is macroscopic and continuous.

We start by specifying the most general constitutive relations that still satisfy the above assumptions. To this end, we write the polarization and magnetization vectors in the form

$$\mathbf{P}(\mathbf{r},t) = \frac{\varepsilon(\mathbf{r})-1}{4\pi}\mathbf{E}(\mathbf{r},t) + \int_{-\infty}^{t}\sigma(\mathbf{r})\mathbf{E}(\mathbf{r},t')dt' \ , \quad \mathbf{M}(\mathbf{r},t) = \frac{\mu(\mathbf{r})-1}{4\pi}\mathbf{H}(\mathbf{r},t) \ . \tag{32}$$

Here $\varepsilon(\mathbf{r})$, $\sigma(\mathbf{r})$ and $\mu(\mathbf{r})$ are purely real functions, which completely characterize the medium in the adopted approximation. Note that the constitutive relation (32) corresponds to the complex frequency-domain permittivity $\epsilon(\mathbf{r},\omega) = \varepsilon(\mathbf{r}) + 4\pi i\sigma(\mathbf{r})/\omega$. Further, equations (32) and (16) imply that the total induced electric charge is

$$\rho_{\rm e}(\mathbf{r},t) = \rho_{\rm e}^{(\rm d)}(\mathbf{r},t) + \rho_{\rm e}^{(\rm c)}(\mathbf{r},t) \ , \tag{33a}$$

where

$$\rho_{\rm e}^{(\rm d)}(\mathbf{r},t) = -\nabla\cdot\frac{\varepsilon(\mathbf{r})-1}{4\pi}\mathbf{E}(\mathbf{r},t) \equiv -\nabla\cdot\mathbf{P}^{(\rm d)}(\mathbf{r},t) \ , \quad \rho_{\rm e}^{(\rm c)}(\mathbf{r},t) = -\nabla\cdot\int_{-\infty}^{t}\sigma(\mathbf{r})\mathbf{E}(\mathbf{r},t')dt' \ . \tag{33b}$$

Here $\rho_{\rm e}^{(\rm d)}$ is the electric charge density associated with the dielectric properties of the medium and $\rho_{\rm e}^{(\rm c)}$ is related to conductivity; we have also introduced the vector $\mathbf{P}^{(\rm d)} \equiv [(\varepsilon-1)/4\pi]\mathbf{E}$, which is the "dielectric" part of the total polarization field $\mathbf{P}$. We emphasize that exactly the same laws apply to both charge densities $\rho_{\rm e}^{(\rm c)}$ and $\rho_{\rm e}^{(\rm d)}$ and to their currents, and that the same force acts upon them; we have written the two terms separately for clarity and to establish a relation to many previous expositions of the subject.

We now proceed with the actual calculation. To start, we write the first term in the right-hand side of (24) in the form

$$\begin{aligned}
\rho_{\rm e}\mathbf{E} &= \rho_{\rm e}^{(\rm c)}\mathbf{E} - (\nabla\cdot\mathbf{P}^{(\rm d)})\mathbf{E} \\
&= \rho_{\rm e}^{(\rm c)}\mathbf{E} + (\mathbf{P}^{(\rm d)}\cdot\nabla)\mathbf{E} - \nabla\cdot(\mathbf{E}\otimes\mathbf{P}) \\
&= \rho_{\rm e}^{(\rm c)}\mathbf{E} + \frac{\varepsilon-1}{4\pi}(\mathbf{E}\cdot\nabla)\mathbf{E} - \nabla\cdot(\mathbf{E}\otimes\mathbf{P}) \ ,
\end{aligned} \tag{34}$$

where $\otimes$ denotes tensor product of two vectors. We rely here on the conventional definition of divergence of a tensor. That is, if $\hat{T}$ is a tensor with Cartesian components $T_{\alpha\beta}$, then $(\nabla\cdot\hat{T})_\alpha = \sum_\beta \partial T_{\alpha\beta}/\partial r_\beta$. Thus, we have expressed the force acting on the induced electric charge as a sum of the "conductivity" and "dielectric" parts, and then transformed the expression $(\nabla\cdot\mathbf{P}^{(\rm d)})\mathbf{E}$ identically to obtain a term of the form (25b) plus a divergence of a tensor. In fact, the existence of such a transformation proves the relations (26), since the volume integral of any divergence vanishes as long as the integrand is zero on the surface bounding the integration region.

We next use the identity

$$(\mathbf{a}\cdot\nabla)\mathbf{a} = (\nabla\times\mathbf{a})\times\mathbf{a} + \frac{1}{2}\nabla a^2 \tag{35}$$

to continue the line of equalities in (34) as

$$\begin{aligned}
\rho_{\rm e}\mathbf{E} &= \rho_{\rm e}^{(\rm c)}\mathbf{E} + \frac{\varepsilon-1}{4\pi}(\nabla\times\mathbf{E})\times\mathbf{E} + \frac{\varepsilon-1}{8\pi}\nabla E^2 - \nabla\cdot(\mathbf{E}\otimes\mathbf{P}) \\
&= \rho_{\rm e}^{(\rm c)}\mathbf{E} + \frac{\varepsilon-1}{4\pi}(\nabla\times\mathbf{E})\times\mathbf{E} - \frac{E^2}{8\pi}\nabla\varepsilon + \nabla\cdot\left(\frac{\varepsilon-1}{8\pi}E^2\hat{I} - \mathbf{E}\otimes\mathbf{P}\right) \ ,
\end{aligned} \tag{36}$$

where $\hat{I}$ is the identity tensor. Analogously, we have

$$\rho_{\rm m}\mathbf{H} = \frac{\mu-1}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H} - \frac{H^2}{8\pi}\nabla\mu + \nabla\cdot\left(\frac{\mu-1}{8\pi}H^2\hat{I} - \mathbf{H}\otimes\mathbf{M}\right) \ , \tag{37}$$

except that in the magnetic case there is no "conductivity" term. Substituting (36) and (37) into (24b) and using (23) for $\mathbf{S}_2$, we obtain:

$$\mathbf{f}_2 = \rho_{\rm e}^{(\rm c)}\mathbf{E} - \frac{E^2}{8\pi}\nabla\varepsilon - \frac{H^2}{8\pi}\nabla\mu + \frac{1}{4\pi}\left[\mu(\nabla\times\mathbf{H})\times\mathbf{H} + \varepsilon(\nabla\times\mathbf{E})\times\mathbf{E}\right] - \frac{1}{4\pi c}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t} + \nabla\cdot\hat{T} \ . \tag{38}$$

Here

$$\hat{T} = \left[\frac{\varepsilon-1}{8\pi}E^2 + \frac{\mu-1}{8\pi}H^2\right]\hat{I} - (\mathbf{E}\otimes\mathbf{P} + \mathbf{H}\otimes\mathbf{M}) \ , \tag{39}$$

and we emphasize again that the term $\nabla\cdot\hat{T}$ does not produce any net force or torque on an isolated body. Also, $\hat{T}$ *is not* the Maxwell's stress tensor.

At the next step we utilize the macroscopic Maxwell's equation's (10) [9] to transform the terms in the square brackets of (38). Specifically, we write

$$\nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = \frac{\varepsilon}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi\sigma}{c}\mathbf{E} \ , \quad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = \frac{\mu}{c}\frac{\partial\mathbf{H}}{\partial t} \ , \tag{40}$$

where we have utilized the constitutive relations (32). Upon substitution of (40) into (38), we arrive at

$$\mathbf{f}_2 = \rho_{\rm e}^{(\rm c)}\mathbf{E} + \frac{\sigma}{c}\mathbf{E}\times\mathbf{H} - \frac{E^2\nabla\varepsilon + H^2\nabla\mu}{8\pi} - \frac{\varepsilon\mu-1}{4\pi c}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t} + \nabla\cdot\hat{T} \ . \tag{41}$$

This is exactly the expression [57, Eq. (75.18)] modulo some terms depending on the derivatives of the constitutive parameters with respect to the mass density [10], the terms related to conductivity (included explicitly in (41) but omitted in [57]), and also the term $\nabla\cdot\hat{T}$, which integrates to zero and is discussed below in more detail.

The first conclusion we can make is that the force density (41) follows directly from the Lorentz force (as defined in the Second Model of magnetization) and the macroscopic Maxwell's equations. The Lorentz force is primary here; equation (19) is the starting point for deriving the force density (41), and (19) does not depend on the form of macroscopic equations at all. Equation (24b) follows from (19) if, in addition, the most general form of macroscopic Maxwell's equations is used. It is true that one must choose the model for magnetization in order to decide whether (11) or (19) should be used as the starting point. But once this choice is made, no additional extraneous considerations are needed. In particular, derivation of (41) does not require invocation of the variational principle, nor the Maxwell's stress tensor. Derivations that utilize these concepts or quantities are, in fact, circular. For example, Maxwell stress tensor is a direct consequence of the Lorentz force and is not mathematically-independent from the latter. Same is true for the variational principle or minimization of energy, or the methods that utilize various definitions of momentum density in continuous media, etc. All these approaches are superfluous.

The second conclusion is that (41) is obtained, essentially, by assuming that the Second Model of magnetization is physically correct. Expressions of the form (41) can not be derived from (24a), which is the force density in the First Model. Of course, the two equations (24a) and (24b) coincide when $\mathbf{M} = 0$. Therefore, many theoretical treatments of the subject simply make the assumption of non-magnetic media and then discuss whether the Abraham's force (31) is "real" or can be measured in an experiment, i.e., see [25]. However, the terms containing time derivatives like (31) are direct consequences of the Lorentz force law and proof of their existence does not require any extrinsic considerations such as defining the momentum density in continuous media and resolving the Abraham-Minkowski dilemma (generally, an impossible task). In [29], it was shown that the conventional Abraham's force follows from the Lorentz force for non-dispersive dielectrics. Here we have generalized this derivation to media with non-dispersive permittivity, permeability and conductivity. The more general expression, however, is (24b), and it does not contain a term of the form (31) at all; the quantities ε and μ can not even be defined in the general case. Equation (24b) does contain the time derivative $\partial\mathbf{E}\times\mathbf{H}/\partial t$, and including this term into consideration is important if we want to describe correctly the center-of-energy motion as is shown below. But the truly consequential choice one faces is not whether

[9] As discussed in Section 2.4, we use the conventional set of macroscopic Maxwell's equations (10) for both models of magnetization and then rely on the correspondence (21).

[10] These so-called electro- and magneto-striction terms can be significant at low frequencies. However, since they do not influence the physical quantities examined in the Balazs thought experiment or the Abraham-Minkowski dilemma, they are commonly excluded from consideration in the related literature.

to include this term or not (obviously, it should be included), but which of the fundamental force laws (11) or (19) [equivalently, (24a) or (24b)] should be used as the starting point.

Third, we make a comment about force densities that integrate to zero. We have collected all such terms in the expression $\nabla \cdot \hat{T}$, and it is clear that we can add an arbitrary tensor to $\hat{T}$ without influencing the total force and torque. However, $\nabla \cdot \hat{T}$ does influence the local distribution of the density of force. Do we have any reasons to believe that the tensor $\hat{T}$ in (41) is correct or should we just set $\hat{T} = 0$ as in [57, Eq. (75.18)]? Unfortunately, there is no general way to answer this question, and many attempts to do so have been inconclusive. Had we replaced $\mathbf{f}_{\mathrm{a}}$ with $\mathbf{f}_{\mathrm{b}}$ in our derivation of (41), the result for $\hat{T}$ would have been different. In fact, if we accept that $\mathbf{f}_{\mathrm{b}}$ is the correct force density in a dielectric (and similarly for is magnetic counter-part), then the derivation would result in $\hat{T} = 0$ and, in addition, the terms

$$-\frac{E^2}{8\pi}\nabla\varepsilon \equiv \mathbf{f}_{\mathrm{c}} \, , \quad -\frac{H^2}{8\pi}\nabla\mu \equiv \mathbf{g}_{\mathrm{c}} \tag{42a}$$

would be replaced by

$$\frac{\varepsilon}{8\pi}\nabla E^2 \equiv \mathbf{f}_{\mathrm{d}} \, , \quad \frac{\mu}{8\pi}\nabla H^2 \equiv \mathbf{g}_{\mathrm{d}} \, . \tag{42b}$$

Which of these forms is correct? The first equation in (42a) was derived in [57] for fluids in mechanical and thermodynamic equilibrium. However, the assumption that a fluid is in equilibrium under the action of electromagnetic forces can hold only in statics. Generalizations of this result beyond the static limit are dubious. Indeed, it is easy to show that the volume integral of $\mathbf{f}_{\mathrm{c}}$ coincides with that of $\mathbf{f}_{\mathrm{a}}$ or $\mathbf{f}_{\mathrm{b}}$ only for potential (irrotational) fields. Recall that the integrals of $\mathbf{f}_{\mathrm{a}}$ and $\mathbf{f}_{\mathrm{b}}$ are the same even if the fields are not potential.

Author's opinion is that (42) or the more general equation (41) should be used with caution beyond the static limit. For instance, consider a non-conducting but weakly absorbing half-space with $\epsilon = 1 + i\delta$ and $\sigma = 0$, where $\delta \ll 1$. Since δ is arbitrarily small, it is tempting to assume that the conditions of applicability of (41) have been fulfilled and one can safely use the real-valued $\varepsilon = 1$ instead of the complex ϵ. It then follows from (41) that an incident monochromatic plane wave does not exert any pressure on the half-space. In reality, the incident wave is totally absorbed in a sufficiently thick slab for any finite value of δ, and in the limit of a half-space the pressure is not zero. In other words, the *total* absorption can be significant even in weakly absorbing media if the propagation depth is large enough. In principle, one can account for this effect by introducing a nonzero conductivity σ. Then, if $\varepsilon + 4\pi i\sigma/\omega$ matches exactly the complex permittivity of the medium $\epsilon(\omega)$ at the working frequency, Eq. 41 predicts the correct pressure on a half-space or on any finite slab. But this works only for a monochromatic field. If the incident field is not monochromatic and the actual dispersion of the medium does not follow the simple law $\varepsilon + 4\pi i\sigma/\omega$, then (41) can easily predict an incorrect result. The more general laws in (24) are free of this problem and applicable to media with arbitrary dispersion, nonlocal and even nonlinear constitutive relations.

3. Balazs thought experiment

The Balazs thought experiment is based on examination of the motion of the center of energy of a system consisting of a transient electromagnetic pulse and a "block", that is, a finite-width slab of some polarizable and, perhaps, magnetizable material. The pulse comes from infinity and is partially reflected, partially absorbed and partially transmitted through the block.

The center of energy is a relativistic generalization of the familiar center of mass of Newtonian mechanics where the rest masses of particles or mass densities of extended bodies are replaced by the total energies or energy densities. Naturally, this definition allows one to include particles of zero rest mass into the calculation and therefore account for the energy of electromagnetic field.

The test that is applied in the Balazs thought experiment is that the center of energy of the system "pulse+block" must move with a constant velocity in any inertial reference frame. More specifically, one expects that the center of energy $\mathbf{R}_{\mathtt{CE}}(t)$ of the above system (or any closed relativistic system) moves according to the law

$$\mathbf{R}_{\mathtt{CE}}(t) = \mathbf{R}_0 + \frac{c^2 \mathbf{P}_{\mathtt{tot}}}{E_{\mathtt{tot}}} t = \mathbf{R}_0 + \mathbf{V}_{\mathtt{CE}} t \, , \tag{43}$$

where $\mathbf{P}_{\mathtt{tot}}$ and $E_{\mathtt{tot}}$ are the total conserved momentum and energy of the system and $\mathbf{V}_{\mathtt{CE}} = c^2 \mathbf{P}_{\mathtt{tot}} / E_{\mathtt{tot}}$ is the center-of-energy velocity. For one particle, this equation is trivial and for a system of non-interacting particles it can

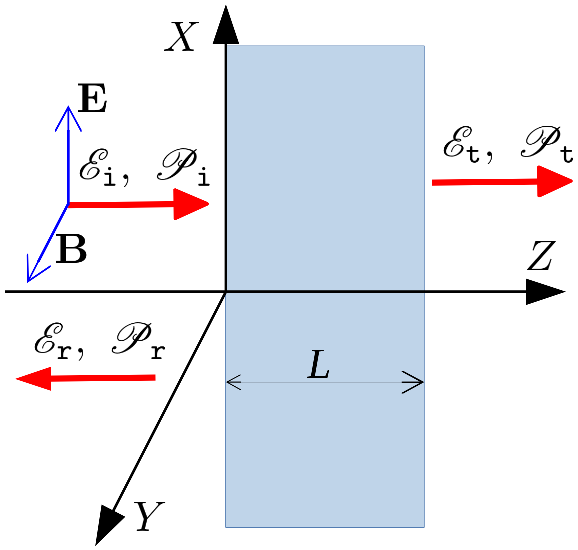

Figure 1: Sketch of the thought experiment considered in this paper.

be obtained as a straightforward generalization. If point particles interact via classical electromagnetic fields under the set of first principles consisting of the microscopic Maxwell's equations, the Lorentz force, and the relativistic laws of motion for the particles, then (43) can also be proved, as was noted by Pryce in 1948 [58]. A detailed derivation of (43) for a set of point particles interacting via classical electromagnetic field was given by Boyer [59, 60]. We note that this derivation can be generalized to the case when both electric and magnetic charges are present and the Lorentz force is given by the expression (4).

One difficulty with the above center-of-energy theorem is that the first principles mentioned above are not consistent with stability of matter. If one introduces some phenomenological non-electromagnetic forces of constraint that hold the system together, a simple law of the form (43) can no longer be proved. This does not mean that (43) is not generally true since the forces of constraint are instantaneous and not consistent with the relativistic physics. A general proof of the center-of-energy theorem for condensed matter interacting with electromagnetic field can probably be obtained in the framework of relativistic quantum mechanics. Author is not aware of any such derivation, and the required theory appears to be very complicated. However, we will proceed below under the assumption that the center-of-energy theorem is generally true and is expected to hold for the Balazs block experiment in particular.

We will indeed find that this is the case for both models of magnetization considered. We will not be able to prove this statement at each moment of time since the interaction of the pulse and the block can be quite complicated. But we will prove that, if $\mathbf{R}_{\rm CE}(t)$ obeyed (43) before the interaction (in the distant past), it will obey the same law after the interaction (in the distant future). Thus, we will develop a *collision theory* for the pulse and the block under some very general assumptions about the properties of these two colliding objects.

The thought experiment analyzed in this paper is illustrated in Fig. 1. A light pulse is incident from left onto a slab of material, which is initially located between the planes $z = 0$ and $z = L$. The pulse is partially reflected, partially absorbed inside the slab, and partially transmitted. We do not make any assumptions about the physical properties of the slab material or the incident pulse, apart from the following:

1. The incident, reflected and transmitted fields are normally incident, linearly polarized plane waves.
2. The slab is not chiral and does not rotate the polarization of the reflected and transmitted waves, which are in this case the same as the polarization of the incident wave as shown by arrows in Fig. 1. Otherwise, we must include angular momentum and rotational energy into consideration, which we wish to avoid.
3. The slab properties can depend on the coordinate z but not on x and y.
4. The slab is not amplifying (active) and can not generate energy. Otherwise, some integrals considered below will not converge and consideration of the conservation laws is problematic.

5. Finally, we assume that the incident pulse is transient and decays sufficiently fast when $t \to \pm\infty$. In particular, the incident pulse carries a finite energy per unit area of the slab. A more precise mathematical condition will be given below.

We note that in the theory presented below the slab can be absorbing, dispersive, and electromagnetically nonlinear. The electric and magnetic susceptibilities can be temporally and spatially non-local. The mass density of the slab, $\rho(z)$, can depend on z. Finally, there is no requirement that the pulses be monochromatic or quasi-monochromatic.

Importantly though, the problem under consideration is one-dimensional. The slab and the wave fronts are viewed as infinite in x and y directions. It is natural, therefore, to define certain physical quantities per unit area of the slab. Below, such quantities will be denoted by curly symbols. For example, $\mathscr{E}_i$ is the total energy per unit area carried by the incident pulse and $\mathscr{P}_i = \mathscr{E}_i/c$ is the incident linear momentum per unit area. Similar quantities can be introduced for the reflected and transmitted waves.

The per unit area mass of the slab is defined by

$$\mathscr{M} = \int_0^L \rho(z) dz \; . \tag{44}$$

We will also need to define the center of energy (same as the center of mass) of the block at rest, Z_0. If the block is homogeneous in the z-direction, then $Z_0 = L/2$. More generally,

$$Z_0 = \frac{1}{\mathscr{M}} \int_0^L z\rho(z) dz \; . \tag{45}$$

Similarly, if the electromagnetic fields exert the force $f_z(z,t)$ per unit volume of the slab material (projected onto the Z-axis), then the force per unit area (pressure) is given by

$$\mathscr{F}(t) = \int_{-\varepsilon}^{L+\varepsilon} f_z(z,t) dz \; . \tag{46}$$

Here we have included an infinitesimally small constant ε in the integration limits to account properly for the possible singularities in the force density (surface terms).

For economy of writing, we will drop in the remainder of this paper the qualifier "per unit area" and refer to the quantities $\mathscr{E}$, $\mathscr{P}$, $\mathscr{M}$ and $\mathscr{F}$ as to energy, momentum, mass and force, respectively.

Finally, in what follows, we will neglect relativistic effects and corrections of higher order. The momentum transfer (from the pulse to the slab) and the slab displacement are effects of the first order. These are very small effects under all practical conditions. However, there exist effects of even higher order, i.e., due to the motion of the slab during the interaction, or due to the relativistic corrections to the law of motion of the slab. These higher-order corrections are much smaller than the first-order quantities. In fact, they are so small that it is safe to say they will never be observed. Although making a complete list of higher-order corrections might be difficult, we can list the following approximations:

1. The optical forces will be computed under the assumption that the slab is at rest.
2. Motion of the slab due to the optical forces will be considered as non-relativistic, i.e., governed by Newton's second law.
3. Kinetic energy of the slab will not be accounted for in the equation expressing energy conservation. However, the momentum transfer will be accounted for. This is typical in problems involving collision of two masses m and M where $M \gg m$. That is, we account for the momentum transfer from the small object to the large one but can disregard the kinetic energy transfer.
4. The mass of the slab in Newton's second law will not change due to absorbed energy. This small re-distribution of mass will create a first-order effect for the motion of the center of energy of the system as a whole, and will be accounted for. But the influence of this additional mass on the motion of the slab is a higher-order (and an incredibly small) effect.
5. Finally, deformations in the slab and elastic forces will be disregarded. This might be not such a good approximation and deformations are not a higher-order relativistic effects. Rather, we assume here that the inter-atomic bonds are very strong relative to the optical forces.

4. Velocity and shift due to a transient pulse of force

At $t \to -\infty$, the slab is at rest. In the process of reflection and transmission, the electromagnetic fields exert some force $\mathscr{F}(t)$ on the slab. It can be expected that the slab will move with some acceleration during the period of interaction. This acceleration depends on the shape of the incident pulse and on the material properties of the slab. However, when all the electromagnetic fields inside the slab die out at $t \to +\infty$, the slab will move with a constant velocity.

Before the block started to move (in the distant past), it was located between the planes $Z_1 = 0$ and $Z_2 = L$. After the interaction (in the distant future) the block will move with a constant velocity and occupy the space between the planes $Z_1(t)$ and $Z_2(t) = Z_1(t) + L$. We can write quite generally for $Z_1(t)$:

$$Z_1(t) = v_\infty t - \Delta_\infty \ , \ \ t \to +\infty \ . \tag{47}$$

To relate the velocity v_∞ and the shift Δ_∞ to the force $\mathscr{F}(t)$, we use the equation of motion

$$\frac{d^2 Z_1(t)}{dt^2} = \frac{1}{\mathscr{M}} \mathscr{F}(t) \ . \tag{48}$$

The solution to (48) is

$$Z_1(t) = \frac{1}{\mathscr{M}} \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} \mathscr{F}(t'') dt'' \ . \tag{49}$$

By changing the order of integration in (49), we can also write

$$Z_1(t) = \frac{1}{\mathscr{M}} \int_{-\infty}^{t} dt'' \mathscr{F}(t'') \int_{t''}^{t} dt' = \frac{t}{\mathscr{M}} \int_{-\infty}^{t} \mathscr{F}(t') dt' - \frac{1}{\mathscr{M}} \int_{-\infty}^{t} t' \mathscr{F}(t') dt' \ . \tag{50}$$

Taking the limit $t \to \infty$ in (50) and comparing to (47), we obtain

$$v_\infty = \frac{1}{\mathscr{M}} \int_{-\infty}^{\infty} \mathscr{F}(t) dt \ , \tag{51a}$$

$$\Delta_\infty = \frac{1}{\mathscr{M}} \int_{-\infty}^{\infty} t \mathscr{F}(t) dt \ . \tag{51b}$$

The result (51a) could have been easily anticipated because it simply states that the total momentum transferred to the slab is equal to the time integral of the force. However, expression (51b) for Δ_∞ is less trivial. It shows that the shift in (47) is uniquely determined by the first time-moment of the force. The shift can be nonzero even if the integral of the force in (51a) is zero and $v_\infty = 0$.

Also, expression (51b) clarifies what was meant above by the condition that the incident pulse is transient. For (47) to hold, it is required that the force falls off when $|t| \to \infty$ faster than $1/|t|^2$. Since the force is quadratic in the electromagnetic fields, the latter should fall off faster than $1/|t|$. We note that such pulses are Fourier-transformable and therefore have well-defined spectra.

5. Incident, reflected and transmitted pulses

We consider an arbitrary incident pulse, requiring only that it satisfies the condition of transiency. The most general plane-wave, linearly-polarized solutions to Maxwell's equations in the half-spaces to the left and to the right of the slab can be written as follows:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} \begin{cases} \psi_{\mathtt{i}}\left(t - \frac{z}{c}\right) + \psi_{\mathtt{r}}\left(t + \frac{z}{c}\right), & z < 0 \\ \psi_{\mathtt{t}}\left(t - \frac{z-L}{c}\right), & z > L \end{cases} \ ; \ \mathbf{B}(z,t) = \hat{\mathbf{y}} \begin{cases} \psi_{\mathtt{i}}\left(t - \frac{z}{c}\right) - \psi_{\mathtt{r}}\left(t + \frac{z}{c}\right), & z < 0 \\ \psi_{\mathtt{t}}\left(t - \frac{z-L}{c}\right), & z > L \end{cases} \tag{52}$$

The subscripts `i`, `r` and `t` indicate the incident, reflected and transmitted fields. Note that in empty space $\mathbf{B} = \mathbf{H}$. The above formulas are written for $\mathbf{B}$. In (52), $\psi_{\mathtt{i}}(t)$ is an arbitrary (modulo the transiency condition) real-valued function, which fully defines the incident pulse. The reflected and the transmitted pulses are determined uniquely by the form of the incident pulse and the properties of the slab. We will not require or rely on a detailed knowledge of these functions, nor will we need to know explicitly the fields inside the slab, which can be quite complicated in the general case.

We will require, however, expressions for the total energy and the time dependence of the center-of-energy position of each pulse at $t \to \pm\infty$ in terms of the functions $\psi_{\mathtt{i}}(t)$, $\psi_{\mathtt{r}}(t)$ and $\psi_{\mathtt{t}}(t)$. To compute these quantities, we use the expression for the density of electromagnetic energy in free space

$$U = \frac{1}{8\pi} \left(\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} \right) \ . \tag{53}$$

For the total energies $\mathscr{E}_\alpha$ carried by each pulse ($\alpha = \mathtt{i}, \mathtt{r}, \mathtt{t}$), we have

$$\mathscr{E}_\alpha = \frac{c}{4\pi} \int_{-\infty}^{\infty} \psi_\alpha^2(t) dt \ , \ \ \alpha = \mathtt{i}, \mathtt{r}, \mathtt{t} \ . \tag{54}$$

The total linear momenta $\mathscr{P}_\alpha$ of the pulses (projected onto the Z-axis) are given by

$$\mathscr{P}_{\mathtt{i}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_{\mathtt{i}}^2(t) dt \ , \ \ \mathscr{P}_{\mathtt{r}} = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_{\mathtt{r}}^2(t) dt \ , \ \ \mathscr{P}_{\mathtt{t}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_{\mathtt{t}}^2(t) dt \ . \tag{55}$$

Of course, these energies and momenta are measurable only at the times when the pulses do not overlap with each other or the slab. In the distant past ($t \to -\infty$), the incident pulse does not overlap or interfere with any other fields or with the slab so that its total energy and momentum are well-defined and can be measured. Likewise, in the distant future ($t \to +\infty$), only the reflected and the transmitted pulses exist; they do not overlap with each other or the slab and their energies and momenta can also be measured.

For the centers of energy $Z_\alpha(t)$ of the incident, reflected and transmitted pulses, we have the following relations:

$$Z_{\mathtt{i}}(t) = \ \ c(t - \tau_{\mathtt{i}}) \ , \ \ t \to -\infty \ , \tag{56a}$$
$$Z_{\mathtt{r}}(t) = -c(t - \tau_{\mathtt{r}}) \ , \ \ t \to +\infty \ , \tag{56b}$$
$$Z_{\mathtt{t}}(t) = \ \ c(t - \tau_{\mathtt{t}}) + L \ , \ \ t \to +\infty \ . \tag{56c}$$

where the time shifts τ_α are defined by the relations

$$\tau_\alpha = \frac{\int_{-\infty}^{\infty} \psi_\alpha^2(t) t dt}{\int_{-\infty}^{\infty} \psi_\alpha^2(t) dt} = \frac{c}{4\pi \mathscr{E}_\alpha} \int_{-\infty}^{\infty} \psi_\alpha^2(t) t dt \ . \tag{57}$$

Note that τ_α are determined by the first moments of the functions $\psi_\alpha(t)$, similarly to the spatial shift Δ_∞ being determined, according to (51b), by the first moment of the force acting on the slab. This is a useful observation, which will allow us to demonstrate the center-of-energy theorem for the system "pulse+block" without a detailed knowledge of the functions $\psi_\alpha(t)$.

In this Section, we relied on the expressions for the energy and momentum density of the electromagnetic field in free space. These expressions are not controversial and can be considered as being well established. However, it should be acknowledged that they are not independent but follow from the generalized force law of the form (4) and microscopic Maxwell's equations. These densities are consistent with the conservation laws for an arbitrary system of relativistic but otherwise classical (that is, not quantum) point particles [60]. The above reference did not introduce magnetic charge (the monopoles), but this generalization is straightforward to make. In the following section, we will use the set-up of the Balazs though experiment to show that the same laws hold in the case of condensed matter and macroscopic rigid bodies.

6. Conservation laws

We are now ready to analyze the Balazs thought experiment. We will consider the three conservation laws (energy, momentum and the center-of-energy theorem) for the two force laws that are defined in equations (22) and (equivalently) in (24).

6.1. Energy

We first establish that both models of magnetization discussed in Sec. 2 conserve energy. To this end we use the expressions for Poynting vector (23). The total energy absorbed by the slab is given by

$$\mathcal{Q} = \int_{-\infty}^{\infty} dt \int_{-\varepsilon}^{L+\varepsilon} dz\, q(z,t) \,, \tag{58}$$

where

$$q(z,t) = -\nabla \cdot \mathbf{S}(z,t) = -\partial S_z(z,t)/\partial z \,. \tag{59}$$

Here $\mathbf{S}$ is given by one of the expressions in (23) and S_z is the z-component of the Poynting vector; note that, in the considered geometry, $S_x = S_y = 0$. Also, ε is an infinitesimally small constant; it is needed to make the integral in (58) well-defined in the cases when $\mathbf{S}$ can jump at the boundary and the term $\nabla \cdot \mathbf{S}$ has a singularity. In cases when there are no such singularities, the inclusion of the parameter ε in the definition (58) does not make any difference. Integration over z can be easily done and we obtain the result

$$\mathcal{Q} = \int_{-\infty}^{\infty} dt [S_z(-\varepsilon,t) - S_z(L+\varepsilon,t)] \,. \tag{60}$$

Equation (60) is quite as expected: the absorbed energy is equal to the energy that came into the slab minus the energy that came out. Also, since we have considered a transient process, all the absorbed energy goes into some form of internal energy of the slab. This can include heat and some other forms of internal energy. But it is *not* electromagnetic energy in the sense of macroscopic electrodynamics: indeed, in the end of the process, the slab is neither polarized nor magnetized, and all macroscopic electromagnetic fields inside the slab die out.

We now use the fact that, in vacuum, $\mathbf{H} = \mathbf{B}$. Since the points $z = -\varepsilon$ and $z = L + \varepsilon$ are in vacuum, we can use without loss of generality the first expression (23) for $\mathbf{S}$ and write

$$\mathcal{Q} = \frac{c}{4\pi} \int_{-\infty}^{\infty} [E_x(-\varepsilon,t) B_y(-\varepsilon,t) - E_x(L+\varepsilon,t) B_y(L+\varepsilon,t)] dt \,, \tag{61}$$

where the fields $E_x(z,t)$ and $B_y(z,t)$ are defined in (52). We now use these definitions to write $\mathcal{Q}$ in terms of the functions $\psi_\alpha(t)$, viz,

$$\mathcal{Q} = \frac{c}{4\pi} \int_{-\infty}^{\infty} [\psi_{\mathtt{i}}^2(t) - \psi_{\mathtt{r}}^2(t) - \psi_{\mathtt{t}}^2(t)] dt \,, \tag{62}$$

and using (54),

$$\mathcal{Q} = \mathcal{E}_{\mathtt{i}} - \mathcal{E}_{\mathtt{r}} - \mathcal{E}_{\mathtt{t}} \,. \tag{63}$$

This equation is the mathematical expression of the law of energy conservation. Specifically, the incident energy that has not been reflected or transmitted is absorbed by the slab. This result is obtained irrespective of which expression in (23) is used for the Poynting vector. Thus, the expression involving the cross product $\mathbf{E} \times \mathbf{B}$ is fully consistent with energy conservation.

We thus see that both models of magnetization of matter are fully consistent with energy conservation. This result is not surprising and follows from the fact that the expression for the Poynting vector in vacuum is unambiguous and the same in both models. Since any finite object can be enclosed in a regular surface drawn entirely in vacuum, the total energy that is absorbed by this object is unambiguous and independent of the expression for the Poynting vector inside the object. This already has been pointed out in [49]. Here we have shown explicitly how energy conservation works for the particular set-up of the Balazs block thought experiment.

Finally, as noted in Point 3 of the list of approximations in the end of Sec. 3, we did not account for the kinetic energy of the slab in the energy balance. The latter is a higher-order relativistic effect, and it is smaller than the individual terms in (63) by the factor of $\sim \mathcal{E}_{\mathtt{i}}/\mathcal{M}c^2$. We can understand the approximation involved here as the limit $\mathcal{M} \to \infty$.

6.2. Momentum

We now consider conservation of momentum. To this end, we will use explicitly the two different expressions for the force density (22). We start by noting that the total momentum in the distant past is

$$\mathscr{P}_{-\infty} = \mathscr{P}_{\mathtt{i}} \ , \tag{64}$$

where $\mathscr{P}_{\mathtt{i}}$ is given in (55). After the interaction, the total momentum is

$$\mathscr{P}_{\infty} = \mathscr{P}_{\mathtt{t}} + \mathscr{P}_{\mathtt{r}} + \mathscr{M} v_{\infty} \ , \tag{65}$$

where v_∞ is defined in (51a) in terms of the force. We will therefore compute v_∞ for both expressions of the force density in (22) and show that it is the same in both cases and, moreover, we will show that $\mathscr{P}_{-\infty} = \mathscr{P}_{\infty}$.

We start by noting that, irrespective of the model used, the terms $\rho_{\mathtt{e}}$ and $\rho_{\mathtt{m}}$ are zero in the geometry of the Balazs block thought experiment. Moreover, the term $\partial(\mathbf{E}{\times}\mathbf{B})/\partial t$ or $\partial(\mathbf{E}{\times}\mathbf{H})/\partial t$, when inserted into (51a) and integrated over time in the infinite limits, yields zero. We have therefore the following result:

$$\mathscr{M} v_{\infty} = -\frac{1}{8\pi} \int_{-\infty}^{\infty} dt \int_{-\varepsilon}^{L+\varepsilon} \left(\frac{\partial E_x^2}{\partial z} + \frac{\partial F_y^2}{\partial z} \right) dz \ , \tag{66}$$

where F_y is either H_y or B_y depending on which expression in (22) is used. The integral, of course, does not depend on this choice. Indeed, integrating over z, we find that

$$\mathscr{M} v_{\infty} = \frac{1}{8\pi} \int_{-\infty}^{\infty} \left[E_x^2(-\varepsilon, t) + B_y^2(-\varepsilon, t) - E_x^2(L+\varepsilon, t) - B_y^2(L+\varepsilon, t) \right] dt \ , \tag{67}$$

where we have used again the fact that in vacuum $\mathbf{B} = \mathbf{H}$. In terms of the functions $\psi_\alpha(t)$, we have

$$\mathscr{M} v_{\infty} = \frac{1}{8\pi} \int_{-\infty}^{\infty} \left[(\psi_{\mathtt{i}}(t) + \psi_{\mathtt{r}}(t))^2 + (\psi_{\mathtt{i}}(t) - \psi_{\mathtt{r}}(t))^2 - 2\psi_{\mathtt{t}}^2(t) \right] dt \ . \tag{68}$$

Opening the brackets and canceling similar terms, we obtain

$$\mathscr{M} v_{\infty} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\psi_{\mathtt{i}}^2(t) + \psi_{\mathtt{r}}^2(t) - \psi_{\mathtt{t}}^2(t) \right] dt \ . \tag{69}$$

From this and (55), we find immediately that

$$\mathscr{M} v_{\infty} = \mathscr{P}_{\mathtt{i}} - \mathscr{P}_{\mathtt{r}} - \mathscr{P}_{\mathtt{t}} \ . \tag{70}$$

We finally substitute this result into (65) and see that $\mathscr{P}_{\infty} = \mathscr{P}_{-\infty} = \mathscr{P}_{\mathtt{i}}$.

Thus, momentum is also conserved in the Balazs block thought experiment irrespective of which expression for the force density is used.

6.3. Center of energy

We will now describe the center-of-energy motion of the system "pulse+block" in the distant past and distant future and show that, regardless of the expression for the force density, the center of energy moves as expected, that is, with a constant velocity.

Let the total conserved energy of the system "pulse+block" be $\mathscr{E}_{\mathtt{tot}}$ and the center of energy position be $Z_{\mathtt{CE}}(t)$. We have in the distant past:

$$\mathscr{E}_{\mathtt{tot}} Z_{\mathtt{CE}}(t) = \mathscr{E}_{\mathtt{i}} Z_{\mathtt{i}}(t) + \mathscr{M} c^2 Z_0 \ , \quad t \to -\infty \ , \tag{71a}$$

and in the distant future (after the collision):

$$\mathscr{E}_{\mathtt{tot}} Z_{\mathtt{CE}}(t) = \mathscr{E}_{\mathtt{r}} Z_{\mathtt{r}}(t) + \mathscr{E}_{\mathtt{t}} Z_{\mathtt{t}}(t) + \mathscr{M} c^2 (Z_0 + \delta_{\mathtt{q}} + v_\infty t - \Delta_\infty) \ , \quad t \to +\infty \ . \tag{71b}$$

We now explain various terms in (71). First, the energies $\mathscr{E}_\alpha$ and the centers of energy $Z_\alpha(t)$ of the incident, reflected and transmitted ($\alpha = \mathtt{i}, \mathtt{r}, \mathtt{t}$) pulses are defined in (54) and (56). The center of energy of the block at rest, Z_0, is defined in (45); this quantity is unimportant and will cancel out early on in the calculations. The final velocity v_∞ and the shift Δ_∞ of the slab due to the action of electromagnetic force $\mathscr{F}(t)$ are discussed in Sec. 4; specific expressions are given in (51). Additionally, there is a term denoted by $\delta_{\rm q}$. This is the shift of $Z_{\rm CE}$ due to the energy that was transferred from the field to the slab. As was shown by Chau and Lezec [42, 43], account for energy transfer is important in the analysis of center-of-energy motion. We have analogously to (45):

$$\mathscr{M}c^2\delta_{\rm q} = \int_{-\infty}^{\infty} dt \int_{-\varepsilon}^{L+\varepsilon} q(z,t) z dz \ , \tag{72}$$

where $q(z,t)$ is the rate of energy transfer per unit time per unit depth as defined in (59). We keep in mind that S_z in (59) is given by one of the expressions in (23). An interesting fact about $\delta_{\rm q}$ is that it can be nonzero even if the total absorbed energy $\mathscr{Q}$ (as defined by (58)) is zero. This takes place, in particular, in the First Model of magnetization: even if the slab is non-absorbing, interaction with the incident pulse results in some redistribution of energy, which must be accounted for.

Now, if the center of energy of the system "pulse+block" moves, as expected, with a constant velocity, the expressions (71a) and (71b) must be equal. The converse is also true. We will show that this is indeed the case regardless of which model of magnetization we use. We note right away that the two models will result in different Δ_∞ and $\delta_{\rm q}$ but the difference $\delta_{\rm q} - \Delta_\infty$ is unaffected.

We start by noticing that both expressions (71a) and (71b) contain a large term $\mathscr{M}c^2 Z_0$ and it can be canceled. Then, using (56) for $Z_\alpha(t)$, we can write the equality that we wish to verify as follows:

$$\mathscr{E}_{\rm i} c(t-\tau_{\rm i}) \stackrel{?}{=} -\mathscr{E}_{\rm r} c(t-\tau_{\rm r}) + \mathscr{E}_{\rm t}[c(t-\tau_{\rm t}) + L] + \mathscr{M}c^2(\delta_{\rm q} + v_\infty t - \Delta_\infty) \ . \tag{73}$$

We next observe that the terms that are proportional to time t on both sides of (73) are equal to each other. This follows from momentum conservation. To see this mathematically, we can group all the terms proportional to t together, divide the sum by c^2 and use (70). We should keep in mind the sign in the relation $\mathscr{P}_{\rm r} = -\mathscr{E}_{\rm r}/c$. We then will see that the terms proportional to t cancel out from (73). Therefore, (73) is equivalent to the following condition:

$$\mathscr{E}_{\rm i} c\tau_{\rm i} + \mathscr{E}_{\rm r} c\tau_{\rm r} + \mathscr{E}_{\rm t}(L - c\tau_{\rm t}) + \mathscr{M}c^2(\delta_{\rm q} - \Delta_\infty) \stackrel{?}{=} 0 \ . \tag{74}$$

We will next compute Δ_∞. To this end we use the definition (51b) and the expression for the force in the form (24). The calculation is very similar to the one in Sec. 6.2, except that now the term $\partial \mathbf{S}/\partial t$ does not integrate to zero. Specifically, we have

$$\mathscr{M}c^2\Delta_\infty = \frac{c^2}{4\pi}\int_{-\infty}^{\infty}\left[\psi_{\rm i}^2(t) + \psi_{\rm r}^2(t) - \psi_{\rm t}^2(t)\right] t dt - \int_0^L dz \int_{-\infty}^{\infty} \frac{\partial S_z(z,t)}{\partial t} t dt \ . \tag{75}$$

As previously, we understand that S_z is given by one of the expressions in (23), depending on the model used. Also, the integrand in (75) does not have any singularities on the surface so that the limits of integration over z have been written simply as 0 and L. We evaluate the first integral in (75) by using (57) and (54). In the last term, we perform integration over time by parts and account for the fact that the fields decay faster than $1/|t|$ at $t \to \pm\infty$. We obtain in this manner

$$\mathscr{M}c^2\Delta_\infty = \mathscr{E}_{\rm i} c\tau_{\rm i} + \mathscr{E}_{\rm r} c\tau_{\rm r} - \mathscr{E}_{\rm t} c\tau_{\rm t} + I \ , \tag{76}$$

where

$$I \equiv \int_0^L dz \int_{-\infty}^{\infty} S_z(z,t) dt \ . \tag{77}$$

We then substitute (76) into (74) and see that all terms of the form $\mathscr{E}_\alpha c\tau_\alpha$ cancel out. Therefore, the condition we need to verify becomes

$$\mathscr{E}_{\rm t} L + \mathscr{M}c^2\delta_{\rm q} - I \stackrel{?}{=} 0 \ . \tag{78}$$

What remains to be done now is compute δ_{q}. To this end, we substitute the definition of q (59) into (72):

$$\mathscr{M}c^2\delta_{\mathrm{q}} = -\int_{-\infty}^{\infty} dt \int_{-\varepsilon}^{L+\varepsilon} \frac{\partial S_z(z,t)}{\partial z} z dz \ , \tag{79}$$

and evaluate the integral over z by parts. This yields

$$\mathscr{M}c^2\delta_{\mathrm{q}} = -\int_{-\infty}^{\infty} \left[(L+\varepsilon)S_z(L+\varepsilon,t) + \varepsilon S_z(-\varepsilon,t)\right] dt + \int_{-\infty}^{\infty} dt \int_0^L S_z(z,t) dz \ . \tag{80}$$

The last term in (80) is just I as defined in (77) with only the order of integration being reversed. Note that $S_z(z,t)$ does not have any singularities at $z=0$ or $z=L$ (it may have discontinuities) so that we replaced the integration limits $-\varepsilon$ and $L+\varepsilon$ by 0 and L for simplicity. This does not entail any loss of precision or generality. Moreover, the term $\varepsilon S_z(-\varepsilon,t)$ is zero since ε is infinitesimally small and, finally, it is easy to see that

$$\int_{-\infty}^{\infty} (L+\varepsilon)S_z(L+\varepsilon,t)dt = L\mathscr{E}_{\mathrm{t}} \ . \tag{81}$$

Therefore

$$\mathscr{M}c^2\delta_{\mathrm{q}} = -L\mathscr{E}_{\mathrm{t}} + I \ . \tag{82}$$

Substituting this result into (78), we see that the left-hand side is zero and the condition indeed holds.

This proves that the center-of-energy of the system "pulse + block" moves linearly and with a constant velocity regardless of the expression for the force one uses in Newton's second law. However, it is important to account for *all* energy when computing the position of the center of energy, including the energy transferred from the field to the slab. Whereas the total transferred energy $\mathscr{Q}$ is the same in both models, its spatial distribution is different. Not accounting for this spatial distribution properly and not accounting for the resulting shift in the position of the center of energy, δ_{q}, led previously to mistakes, in particular, in [37].

6.4. Non-absorbing slab

The original analysis by Balazs [40], as well as that by Mansuripur [37] relied on the assumption that the slab is non-absorbing. It is instructive therefore to consider this case separately.

We will say that the block is non-absorbing if the total energy transferred from the field to the slab, $\mathscr{Q}$, is zero. This condition is not as strong as $q(z,t) = 0$. In principle, one can have $q(z,t) \neq 0$ but $Q = 0$. This can happen, in particular, in the First Model of magnetization. One commonly recognized case of a non-absorbing medium is a medium characterized by constant and real coefficients ϵ and μ. We emphasize however that the general results of this paper do not assume linearity of the constitutive relations.

Now assume that the block is non-absorbing according to the above definition. In the Second Model of magnetization, Poynting vector is given by the second expression in (23). Since the tangential components of the **E** and **H**-fields are continuous in the conventional formulation of macroscopic Maxwell's equations, the normal component of Poynting vector, S_z, is also continuous. Under these conditions, the integral $\int_{-\infty}^{\infty} S_z(z,t)dt$ can not depend on z and, moreover,

$$\int_{-\infty}^{\infty} S_z(z,t)dt = \mathscr{E}_{\mathrm{i}} - \mathscr{E}_{\mathrm{r}} = \mathscr{E}_{\mathrm{t}} \ . \tag{83}$$

From this property and the definition (77), we find that

$$I = L\mathscr{E}_{\mathrm{t}} \ . \tag{84}$$

If we substitute this result into (78) and assume that $\delta_{\mathrm{q}} = 0$, we will see that the condition holds. Indeed, in case of a non-absorbing medium treated theoretically within the framework of the Second Model, the choice $\delta_{\mathrm{q}} = 0$ is correct. One can, in principle, forget about this quantity and verify the correctness of the center-of-motion theorem.

However, in the First Model, we must use the first expression in (23) for $S_z(z,t)$. In this case, the integral $\int_{-\infty}^{\infty} S_z(z,t)dt$ can depend on z even in a non-absorbing slab. In the simplest case of a homogeneous slab characterized by real-valued constant parameters ϵ and μ, this integral jumps at the surface. It is easy to see that in this case, instead of (84), we would obtain

$$I = \mu L \mathscr{E}_{\mathrm{t}} \,. \tag{85}$$

If we substitute this result into (78) and still assume (now incorrectly) that $\delta_{\mathrm{q}} = 0$, we will obtain a contradiction to the center-of-energy theorem: the left-hand side of (78) will evaluate to $(1-\mu)L\mathscr{E}_{\mathrm{t}} \neq 0$. However, accounting properly for the non-zero δ_{q} will restore the equality in (78) and consistency with the center-of-energy theorem.

7. Summary and discussion

We have shown that both force laws defined in (22) or (equivalently) in (24), which include the conventional Lorentz force [(22a) or (24a)] and the generalized Lorentz (Einstein-Laub) force [(22b) or (24b)], are consistent with conservation of energy, conservation of momentum and the center-of-energy theorem. Einstein-Laub force is however based on the premise that magnetization of matter is caused by displacement of hypothetical magnetic monopoles. The above result was obtained theoretically with minimal assumptions. It does not imply that (24a) and (24b) predict the same *total* force on the slab at each moment of time; this is not so. We have shown that the two forces result in different *geometrical shifts* of the slab $-\Delta_\infty$ after transmission of a finite-energy pulse. The two forces also yield different distributions of the absorbed energy and, consequently, different *energy shifts*, δ_{q}. We also found that the difference $\delta_{\mathrm{q}} - \Delta_\infty$ is the same for the two forces. As a result, the center of energy of the system "pulse+block" moves linearly, just as expected. Note that, if we allow the heat or other form of internal energy to diffuse inside the block and achieve an equilibrium distribution, the resulting reaction force will equalize the geometrical shifts produced by the two forces after the relaxation process is complete.

This result contradicts some previously published claims that favor the Einstein-Laub form of the force density [(22b) or (24b)]. We have identified the sources of errors in these claims. We have shown that, if we treat the problem consistently starting from the correct first principles and not rely on any *ad hoc* approximations, both forces are consistent with all conservation laws. This conclusion is not surprising and could have been glimpsed already after writing out the equation (4) of the microscopic theory. Indeed, the macroscopic electrodynamics is derived from its microscopic counter-part not in some mysterious way but simply by assuming that the current and density of the charges (either electric or magnetic) are continuous and supported inside the medium. This is, in fact, *all* that is involved. The rest follows from symmetries, assumption of linearity of the constitutive relations, etc. However, the transition from the distributions (2) to continuous densities of charge and current is not a big step; the resultant macroscopic theory inherits all the conservation laws of its more fundamental microscopic counter-part.

An important point is that the Second Model of magnetization, while consistent with all conservation laws, is theoretically applicable only to magnetization due to alignment of the intrinsic magnetic moments of elementary particles, that is, to ferro- and para-magnetism. It is not applicable to diamagnetism nor to the "artificial" magnetism, which is obtained by means of electromagnetic homogenization in composite materials. In the latter two cases, there is no doubt that the magnetization is caused by currents of electric charges and introduction of magnetic monopoles into the theory is superfluous. As for ferro- and para-magnetism, these are, generally, low-frequency phenomena. The case $\epsilon = \mu > 0$ and $\lambda \ll L$ (where λ is the characteristic wavelength of the pulse inside the material) is hardly realizable in such materials. Also, we should keep in mind the existence of magnetic domains, which can result in spatial nonlocality of the ferromagnetic response. All this makes the original assumption of local and positive ϵ and μ not very realistic.

One can ask why the Einstein-Laub form of the force density projects such a strong appeal in the physics community. After all, the "conventional expression" (41) is just a special case of the Einstein-Laub formula. In [57] the formula is derived without mentioning this fact or invoking magnetic monopoles (and in a rather indirect way), but it does follow mathematically from the Einstein-Laub force (24b) and *does not* follow from the more conventional expression (24a). The explanation perhaps is that the Second Model of magnetization is more intuitive and symmetric. It treats magnetic and electric polarization on exactly the same footing – as displacement of charges. The fields $\mathbf{E}$ and $\mathbf{H}$ enter the theory in a completely symmetric manner in the Second Model. In addition, the Second Model is consistent with many implicit assumptions that typical textbooks on classical electrodynamics take for granted but never prove. One of these assumptions is that electric field can not do work on the surface currents and therefore the normal component of the Poynting vector must be continuous at magnetic/nonmagnetic interfaces. The price paid for this assumption is the implicit acknowledgment that, after all, magnetic part of the Lorentz force can do nonzero work (on moving electric charges), even though the same textbooks routinely deny this. Thus, in their theoretical treatment of work, force and energy, the conventional textbooks tacitly accept the Second Model of magnetization as generally correct. But the First Model is just as non-contradictory and, in addition, it does not require the existence of magnetic monopoles.

Perhaps, the most important take-home message of this paper is that classical electrodynamics should be taken at its face value. A current is a current, and the Lorentz force acts on it regardless of the current's origin, if it can even be always specified. Partitioning the current into the bound and free or magnetization and polarization components is sometimes possible mathematically but is not necessary or useful physically, and therefore can result in confusion. The induced current $\partial \mathbf{P}/\partial t + c\nabla\times\mathbf{M}$ is not "effective" but ordinary electric current, no different from the conductivity current, and the Lorentz force acts on it as it would on the latter. Same is true for the induced electric charge. There is no great significance in distinguishing between bound and free charges; they are all subject to the same Lorentz force density $\rho_{\rm e}\mathbf{E}$. Therefore, the microscopic Lorentz force and Maxwell's equations form a set of first principles that are sufficient for understanding all macroscopic electromagnetic phenomena. The only necessary phenomenological concepts in classical electrodynamics of continuous media are the forces of constraint that hold the matter together and the introduction of thermal or any other non-electromagnetic energy that is exchanged with the field whenever work is done on the induced currents. Once this is accepted, no other postulate or extrinsic consideration is required.

The above conclusion applies fully to the Abraham-Minkowski controversy. One does not really need to know what the momentum density in continuous media is, and defining such a quantity is not even possible since only the total momentum is physically significant. Rather, all measurable effects such as force, torque and displacement are predictable from the first principles mentioned above. The only meaningful choice that needs to be made is between the two competing models of magnetization of matter (Amperian loops versus displaceable magnetic poles). This has nothing to do with the Abraham-Minkowski controversy. The choice is related to the physical origin of the intrinsic magnetic moments of elementary particles. It can not be made by playing with Maxwell's equations or the conservation laws but requires experimental evidence of the kind that, at present, is far beyond our reach.

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